Algebraic Topology

(1) For $n \geq 3$, we put

$$X_n = \{z \in \mathbb{C} \mid |z| = 1 \text{ or } z^n \in (0, \infty)\}$$
$$Y_n = \{z \in \mathbb{C} \mid |z| = 1 \text{ or } z^n \in [0, \infty)\}.$$

(a) Sketch X_3 and Y_3. **(2 marks)**

(b) Define the terms *homotopy* and *homotopy equivalent*. **(5 marks)**

(c) Prove (by constructing explicit maps and homotopies, and checking their validity) that X_n and X_m are homotopy equivalent for all $n, m \geq 3$. **(8 marks)**

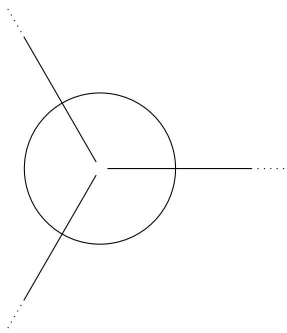
(d) Prove that for all $n \neq m$, the space X_n is not homeomorphic to X_m. **(6 marks)**

(e) Prove that for all $n \neq m$, the space Y_n is not homotopy equivalent to Y_m. **(4 marks)**

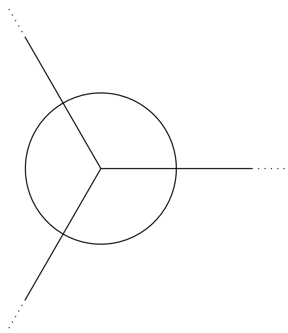
Claims about the homology of particular spaces should be stated clearly and justified briefly, but details are not required.

Solution: This has many ideas in common with Q1 from 2018-19

(a) The spaces X_3 and Y_3 are as follows:



X_3 [1]



Y_3 [1]

(b) **Bookwork** Let A and B be topological spaces. If p and q are continuous maps from A to B, then a *homotopy* from p to q is a continuous map $h\colon [0,1] \times A \to B$ such that $h(0, a) = p(a)$ and $h(1, a) = q(a)$ for all $a \in A$ [2]. We say that A and B are *homotopy equivalent* if there exist continuous maps $A \xrightarrow{f} B \xrightarrow{g} A$ and a homotopy from $g \circ f$ to id_A and a homotopy from $f \circ g$ to id_B. [3]

(c) For $z \in X_p$ we have $z^p \neq 0$ so $z \neq 0$ so it is legitimate to divide by $|z|$. We can therefore define $f\colon X_n \to X_m$ and $g\colon X_m \to X_n$ by $f(z) = z/|z| \in S^1 \subset X_m$ and $g(w) = w/|w| \in S^1 \subset X_n$ [4]. For $z \in X_n$ we have $g(f(z)) = z/|z|$. If z lies on the unit circle then $g(f(z)) = z$. If z lies on one of the rays of X_n then $g(f(z))$ lies on the same ray so the straight line from z to $g(f(z))$ is wholly contained in X_n. It follows that $g \circ f$ is homotopic to the identity by a linear homotopy $h(t, z) = (1 - t)z + t\, z/|z|$ [3]. The same argument shows that $f \circ g$ is homotopic to the identity, so f and g are homotopy equivalences [1].

(d) Say that a point is *special* if its removal separates the space into three path components [2]. The space X_n has precisely n special points, namely the points $e^{2k\pi i/n}$ for $0 \leq k < n$ [2]. If $n \neq m$ then X_n and X_m have different numbers of special points, so they are not homeomorphic [2].

(e) The space Y_n is path connected and has n holes, so $H_1(Y_n) \simeq \mathbb{Z}^n$ [2]. If $n \neq m$ then $H_1(Y_n)$ is not isomorphic to $H_1(Y_m)$, so Y_n cannot be homotopy equivalent to Y_m [2].

(2)

(a) Define what is meant by a *topology* on a set X. **(3 marks)**

(b) What does it mean to say that a topological space X is *Hausdorff*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(c) What does it mean to say that a topological space X is *compact*?
(If your definition relies on any other concepts, then you should define them.) **(3 marks)**

(d) Let X and Y be topological spaces, and let $f\colon X \to Y$ be a continuous injective map. For each of the claims below, give a proof or a counterexample with justification.

  (i) If X is Hausdorff, then Y must also be Hausdorff. **(4 marks)**

  (ii) If X is compact, then Y must also be compact. **(4 marks)**

  (iii) If Y is Hausdorff, then X must also be Hausdorff. **(4 marks)**

  (iv) If Y is compact, then X must also be compact. **(4 marks)**

Solution:

(a) **Bookwork** A *topology* on X is a family τ of subsets of X (called open sets) such that

  (1) The empty set and the whole set X are open [1]

  (2) The union of any family of open sets is open [1]

  (3) The intersection of any finite list of open sets is open. [1]

(b) **Bookwork** Let X be a topological space. Given $a, b \in X$ with $a \neq b$, a *Hausdorff separation* for (a, b) is a pair of open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$ and $U \cap V = \emptyset$[2]. We say that X is *Hausdorff* if every pair of distinct points has a Hausdorff separation [1].

(c) **Bookwork** Let X be a topological space. An *open cover* of X is a family $(U_i)_{i \in I}$ of open sets whose union is all of X [1]. Given such a cover, a *finite subcover* is a subfamily $(U_i)_{i \in J}$ where $J \subseteq I$ is finite and the union is still all of X [1]. We say that X is *compact* if every open cover has a finite subcover [1].

(d) **Unseen**

  (i) Let X be empty, take $Y = \{0, 1\}$ with the indiscrete topology, and let $f\colon X \to Y$ be the inclusion. Then f is continuous and injective and X is (vacuously) Hausdorff but Y is not Hausdorff (because there is no Hausdorff separation for the pair $(0, 1)$). [4]

  (ii) Let X be empty, take $Y = \mathbb{Z}$ with the discrete topology, and let $f\colon X \to Y$ be the inclusion. Then f is continuous and injective and X is compact but Y is not compact (because the open cover by singletons has no finite subcover). [4]

  (iii) Suppose that Y is Hausdorff; we will show that X is also Hausdorff. Suppose that $a, b \in X$ with $a \neq b$. Then $f(a), f(b) \in Y$, with $f(a) \neq f(b)$ because f is injective. As Y is Hausdorff, we can choose disjoint open sets $U, V \subseteq Y$ with $f(a) \in U$ and $f(b) \in V$. As f is continuous, the sets $f^{-1}(U), f^{-1}(V) \subseteq X$ are open. As $f(a) \in U$ and $f(b) \in V$ we have $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. As U and V are disjoint, we have $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Thus, the pair $(f^{-1}(U), f^{-1}(V))$ is a Hausdorff separation for (a, b). [4]

  (iv) Take $X = (0, 1)$ and $Y = [0, 1]$ and let $f\colon X \to Y$ be the inclusion. Then f is continuous and injective. It is standard that subsets of $\mathbb{R}$ are compact iff they are bounded and closed, so Y is compact but X is not. [4]

(3)

(a) Define the terms *chain complex*, *chain map* and *chain homotopy*. **(8 marks)**

(b) Prove that if two chain maps are chain homotopic, then they have the same effect on homology groups. **(5 marks)**

(c) Consider the chain complex T with $T_i = \mathbb{Z}/8$ for all i and $d(x) = 4x$ for all x. Find the homology groups of T. **(3 marks)**

(d) Suppose we have a short exact sequence $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ of chain complexes and chain maps. Suppose that for all $i \in \mathbb{Z}$ we have $H_{2i+1}(A) = H_{2i+1}(C) = 0$ and $|H_{2i}(A)| = 3$ and $|H_{2i}(C)| = 5$. Prove that all homology groups of B are cyclic or trivial, and determine their orders. **(5 marks)**

(e) Let U_* be a chain complex in which all the differentials d_{2i} (for all $i \in \mathbb{Z}$) are surjective homomorphisms. What can we conclude about the homology groups of U_*? **(4 marks)**

Solution:

(a) **Bookwork**

(1) A *chain complex* is a sequence of abelian groups U_i (for $i \in \mathbb{Z}$) [1]equipped with homomorphisms $d_i: U_i \to U_{i-1}$ [1]satisfying $d_{i-1} \circ d_i = 0: U_i \to U_{i-2}$ for all i (or more briefly, $d^2 = 0$). [1]

(2) Let U_* and V_* be chain complexes. A *chain map* from U_* to V_* is a sequence of homomorphisms $f_i: U_i \to V_i$ [1]such that $d_i \circ f_i = f_{i-1} \circ d_i: U_i \to V_{i-1}$ for all $i \in \mathbb{Z}$ (or more briefly, $df = fd$) [1].

(3) Let $f, g: U_* \to V_*$ be chain maps [1]. A *chain homotopy* between f and g is a sequence of homomorphisms $s_i: U_i \to V_{i+1}$ [1]with $ds + sd = g - f$ [1].

(b) **Bookwork** Suppose we have chain maps $f, g: U_* \to V_*$ and a chain homotopy s as above. Consider an element $u \in H_n(U)$, so $u = [z]$ for some cycle $z \in U_n$ with $d(z) = 0$ [1]. As s is a chain homotopy from f to g, we have $g(z) - f(z) = d(s(z)) + s(d(z))$ [1]. As $d(z) = 0$ this becomes $g(z) - f(z) = d(s(z)) \in \text{img}(d)_n = B_n(V)$ [1], so the cosets $[g(z)] = g(z) + B_n(V)$ and $[f(z)] = f(z) + B_n(V)$ are the same [1], or in other words $g_*(u) = f_*(u)$ as required [1].

(c) **Similar examples have been seen** We can identify $\mathbb{Z}/8$ with $\{0, 1, 2, \ldots, 7\}$. The map d sends $0, 2, 4$ and 6 to 0 and 1, 3, 5 and 7 to 4. It follows that $Z_i(T) = \{0, 2, 4, 6\}$ [1]and $B_i(T) = \{0, 4\}$ [1]so the quotient $H_i(T) = Z_i(T)/B_i(T)$ has order $4/2 = 2$ and is isomorphic to $\mathbb{Z}/2$ [1].

(d) **Unseen** Let $A_* \xrightarrow{i} B_* \xrightarrow{p} C_*$ be as described. The Snake Lemma then gives exact sequences

$$0 = H_{2i+1}(A) \to H_{2i+1}(B) \to H_{2i+1}(C) = 0.$$

As the two outer groups are zero the middle one is also zero, so $H_{2i+1}(B) = 0$ [2]. We also have exact sequences

$$H_{2i+1}(C) \to H_{2i}(A) \to H_{2i}(B) \to H_{2i}(C) \to H_{2i-1}(A)$$

The two outer groups are zero, so the middle three groups form a short exact sequence. As $|H_{2i}(A)| = 3$ and $|H_{2i}(C)| = 5$ it follows that $|H_{2i}(B)| = 15$. Up to isomorphism, the only abelian group of order 15 is $\mathbb{Z}/3 \oplus \mathbb{Z}/5$, and this is isomorphic to $\mathbb{Z}/15$ by the Chinese Remainder Theorem (as 3 and 5 are coprime). It follows that $H_{2i}(B) \simeq \mathbb{Z}/15$ [3].

(e) **Unseen** Let U_* be a chain complex in which $d_{2i}: U_{2i} \to U_{2i-1}$ is always surjective, so $B_{2i-1}(U) = U_{2i-1}$. This means that every element $u \in U_{2i-1}$ can be expressed as $u = d(u')$ for some u', so $d(u) = d^2(u') = 0$ [1]. Thus, the homomorphism $d_{2i-1}: U_{2i-1} \to U_{2i-2}$ is zero. We now have $Z_{2i-1}(U) = B_{2i-1}(U) = U_{2i-1}$, so $H_{2i-1}(U) = U_{2i-1}/U_{2i-1} = 0$ [1]. We also have $B_{2i}(U) = 0$ [1]and so $H_{2i}(U) \simeq Z_{2i}(U) = \ker(d_{2i}: U_{2i} \to U_{2i-1})$ [1].

(4) For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A continuous injective map $i: X \to Y$ such that the map $i_*: H_2(X) \to H_2(Y)$ is not injective. **(5 marks)**

(b) A continuous surjective map $p: X \to Y$ such that the map $p_*: H_2(X) \to H_2(Y)$ is not surjective. **(5 marks)**

(c) A contractible space X and a homeomorphism $f: X \to X$ with no fixed points. **(5 marks)**

(d) A continuous injective map $f: S^1 \to S^3$ such that $S^3 \setminus f(S^1)$ is homotopy equivalent to S^1. **(5 marks)**

(e) A continuous injective map $f: S^1 \to S^3$ such that $S^3 \setminus f(S^1)$ is contractible. **(5 marks)**

Solution:

(a) **Similar examples have been seen** Let i be the inclusion $S^2 \to B^3$ [3]. This is continuous and injective, but $H_2(S^2) \simeq \mathbb{Z}$ and $H_2(B^3) = 0$ so the map $i_* \colon H_2(S^2) \to H_2(B^3)$ is zero and is not injective [2].

(b) **Unseen** Let $p\colon [0,1]^2 \to T = S^1 \times S^1$ be the usual gluing map, given by $p(s,t) = (e^{2\pi is}, e^{2\pi it})$ [3]. This is continuous and surjective, but $H_2([0,1]^2) = 0$ and $H_2(T) \simeq \mathbb{Z}$ so the map $p_* \colon H_2([0,1]^2) \to H_2(T)$ is zero and is not surjective [2].

(c) **Similar examples have been seen** Take $X = \mathbb{R}$ and define $f\colon X \to X$ by $f(x) = x+1$ [3]. Then X is contractible and f is a homeomorphism (with $f^{-1}(x) = x - 1$) and f has no fixed points [2].

(d) **Bookwork** Let $f\colon S^1 \to S^3$ be the standard inclusion given by $f(u,v) = (u,v,0,0)$, and put $X = S^3 \setminus f(S^1)$ [2]. We then have

$$X = \{(u,v,w,x) \in \mathbb{R}^4 \mid u^2+v^2+w^2+x^2 = 1,\ (w,x) \neq (0,0)\}.$$

We can thus define $i\colon S^1 \to X$ and $r\colon X \to S^1$ and $h\colon [0,1] \times X \to X$ by $i(w,x) = (0,0,w,x)$ and $r(u,v,w,x) = (w^2+x^2)^{-1/2}(w,x)$ and

$$h(t,(u,v,w,x)) = (t^2u^2 + t^2v^2 + w^2 + x^2)^{-1/2})(tu, tv, w, x).$$

We then find that $r \circ i = \mathrm{id}$ and h gives a homotopy between $i \circ r$ and the identity so i is a homotopy equivalence [3].

(e) **Immediate consequence of bookwork** The generalised Jordan Curve Theorem says that for any continuous injective map $f\colon S^1 \to S^3$, the complement $S^3 \setminus f(S^1)$ has the same homology as S^1 [3] and so cannot be contractible [2].

(5) Put $X = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\}$, so X is homeomorphic to S^3. Put $\omega = e^{2\pi i/3} \in \mathbb{C}$, so $\omega^3 = 1$. Define an equivalence relation on X by $(x,y) \sim (x',y')$ iff $(x',y') = \omega^k(x,y)$ for some k. Put

$$\begin{aligned} Y &= X/\sim \\ U &= \{[x,y] \in Y \mid x \neq 0\} \\ V &= \{[x,y] \in Y \mid y \neq 0\}. \end{aligned}$$

You may assume that U and V are open in Y and that $Y = U \cup V$.

(a) Show that the formula $f([x,y]) = (x^3/|x|^3, y/x)$ gives a well-defined and continuous map $f\colon U \to S^1 \times \mathbb{C}$. Do not assume any properties of the given formula without checking them. **(6 marks)**

(b) Show that f is actually a bijection and that the inverse satisfies

$$f^{-1}(u,z) = \left[(v, z\,v)/\sqrt{1+|z|^2}\right]$$

where v is any one of the three cube roots of u. Do not assume any properties of the given formula without checking them. **(6 marks)**

(c) You may assume without proof that the map $f^{-1}\colon S^1 \times \mathbb{C} \to U$ is also continuous, so f is a homeomorphism. What can you conclude about the homeomorphism type of $U \cap V$? **(3 marks)**

(d) The facts proved for U have obvious counterparts for V; you can assume these without proof. Deduce descriptions of $H_*(U)$, $H_*(V)$ and $H_*(U \cap V)$. **(5 marks)**

(e) Use the Mayer-Vietoris sequence to compute $H_*(Y)$. You should be able to compute $H_k(Y)$ for $k = 0$ and $k \geq 3$. For $k = 1, 2$ you will need to determine a map in the Mayer-Vietoris sequence, which is possible but not so easy. If you cannot see how to do it then you should guess, and give an answer based on your guess. **(5 marks)**

Solution: Somewhat similar examples have been seen

(a) For $(x,y) \in U'$ we have $x \neq 0$ so it is meaningful to define $f_0(x,y) = (x^3/|x|^3,\ y/x) \in \mathbb{C}^2$ [1]. Basic complex analysis shows that this gives a continuous map $f_0\colon U' \to \mathbb{C}^2$ [1]. As $|x^3/|x|^3| = |x|^3/|x|^3 = 1$ we see that $f_0(x,y) \in S^1 \times \mathbb{C}$ [1]. If $(x,y) \sim (x',y')$ then $(x',y') = (\omega^k x, \omega^k y)$ for some k so

$$f_0(x',y') = (\omega^3 x^3/|\omega x|^3, (\omega y)/(\omega x)) = (x^3/|x|^3,\ y/x) = f_0(x,y).\text{[1]}$$

This proves that f_0 is saturated, so it induces a well-defined and continuous map $f\colon U = U'/\sim \to S^1 \times \mathbb{C}$ given by $f([x,y]) = f_0(x,y) = (x^3/|x|^3,\ y/x)$ [2].

(b) Consider a point $(u,z) \in S^1 \times \mathbb{C}$. Let v be a cube root of u, so $|v| = 1$. Put $x = v/\sqrt{1+|z|^2}$ and $y = zx = zv/\sqrt{1+|z|^2}$. We then have $x \neq 0$ and

$$|x|^2 + |y|^2 = (1+|z|^2)^{-1}(|v|^2 + |z|^2|v|^2|) = 1,$$

so $(x,y) \in X$. We also have $x/|x| = v$ and $y/x = (zx)/x = z$ so

$$f([x,y]) = f_0(x,y) = (v^3, z) = (u,z).$$

This proves that f is surjective [3].

Now suppose we have another element $[x', y'] \in U$ with $f([x',y']) = (u,z)$, so $(x'/|x'|)^3 = u$ and $y'/x' = z$. This gives $y' = zx'$ so $(1+|z|^2)|x'|^2 = |x'|^2 + |y'|^2 = 1$ so $|x'| = (1+|z|)^{-1/2} = |x|$. Together with the relation $(x/|x|)^3 = u = (x'/|x'|)^3$ this gives $(x'/x)^3 = 1$, so $x' = \omega^k x$ for some k. This in turn gives $y' = zx' = \omega^k zx = \omega^k y$, so $(x', y') = \omega^k(x,y)$, so $[x', y'] = [x,y]$. This shows that f is also injective, and therefore bijective [3].

(c) As $f\colon U \to S^1 \times \mathbb{C}$ is a homeomorphism, it also gives a homeomorphism from $U \cap V$ to $f(U \cap V)$ [1]. From the formulae in (b) and (c) we see that $f^{-1}(u,z)$ lies in $U \cap V$ iff $z \neq 0$, so f gives a homeomorphism from $U \cap V$ to $S^1 \times \mathbb{C}^\times$ [2].

(d) We now see that U is homotopy equivalent to S^1, and V is also homotopy equivalent to S^1 by a symmetric argument [1]. On the other hand, $U \cap V$ is homotopy equivalent to $S^1 \times S^1$ [1]. This gives $H_0(U) = H_0(V) = H_0(U \cap V) = \mathbb{Z}$ and $H_1(U) = H_1(V) = \mathbb{Z}$ and $H_1(U \cap V) = \mathbb{Z}^2$ and $H_2(U \cap V) = \mathbb{Z}$ [3].

(e) It is easy to see that U, V, $U \cap V$ and Y are all path connected, so $H_0(Y) = \mathbb{Z}$. The interesting parts of the truncated Mayer-Vietoris sequence are now

$$0 \to H_3(Y) \xrightarrow{\delta} H_2(U \cap V) = \mathbb{Z} \to 0$$ [1]

and

$$0 \to H_2(Y) \xrightarrow{\delta} \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^2 \xrightarrow{\beta} H_1(Y) \to 0.$$ [1]

From the first of these we get $H_3(Y) \simeq \mathbb{Z}$ [1](and similar arguments give $H_n(Y) = 0$ for $n > 3$) [1]. One can check that α has the form $(i,j) \mapsto (i, -i-3j)$ so it is injective with image of index 3 in $\mathbb{Z}^2$; this gives $H_2(Y) = 0$ and $H_1(Y) = \mathbb{Z}/3$ [1].