Algebraic Topology

(1)

(a) Explain the terms *homeomorphism* and *homeomorphic*. **(3 marks)**

(b) Explain the terms *homotopy*, *homotopic* and *homotopy equivalent*, distinguishing carefully between them. **(5 marks)**

(c) Consider the following spaces:

$$\begin{aligned} X_0 &= \{z \in \mathbb{C} \mid \mathrm{Re}(z) \notin \mathbb{Z}\} \\ X_1 &= \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in \mathbb{Z}\} \\ X_2 &= \{z \in \mathbb{C} \mid z \notin \mathbb{Z}\} \\ X_3 &= \{z \in \mathbb{R} \mid z \notin \mathbb{Z}\} \\ X_4 &= \{z \in \mathbb{C} \mid |z| \in \mathbb{Z}\}. \end{aligned}$$

(i) Sketch all these spaces. **(5 marks)**

(ii) For which pairs (i, j) is X_i homotopy equivalent to X_j? Justify your answer briefly. In cases where X_i is homotopy equivalent to X_j you should explain why, and in cases where X_i is not homotopy equivalent to X_j, you should explain that as well. **(6 marks)**

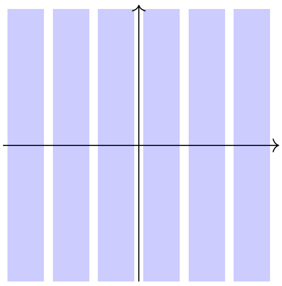
(iii) For which pairs (i, j) is X_i homeomorphic to X_j? Justify your answer briefly. In cases where X_i is homeomorphic to X_j you should explain why, and in cases where X_i is not homeomorphic to X_j, you should explain that as well. **(6 marks)**

Solution:

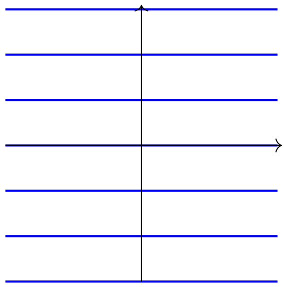
(a) **Bookwork** Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective map $f: X \to Y$ such that both f and the inverse map $f^{-1}: Y \to X$ are continuous [2]. We say that X and Y are *homeomorphic* if there exists such a homeomorphism [1].

(b) **Bookwork** Again let X and Y be topological spaces. Given continuous maps $f_0, f_1: X \to Y$, a *homotopy* from f_0 to f_1 is a continuous map $h: [0,1] \times X \to Y$ with $h(0,x) = f_0(x)$ and $h(1,x) = f_1(x)$ for all $x \in X$ [2]. We say that f_0 and f_1 are *homotopic* if there exists such a homotopy [1]. We say that X and Y are *homotopy equivalent* if there exist maps $f: X \to Y$ and $g: Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y [2].

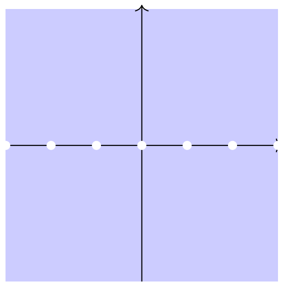
(c) (i) **Similar examples seen** The spaces X_i can be sketched as follows [5]:



X_0

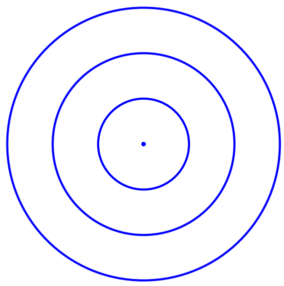


X_1



X_2

X_3



X_4

(ii) **Similar examples have been seen, but this is a bit harder than most of them.**
The spaces X_0, X_1 and X_3 are all homotopy equivalent to $\mathbb{Z}$ and thus to each other. [1]Indeed, we can define maps $\mathbb{Z} \xrightarrow{f_i} X_i \xrightarrow{g_i} \mathbb{Z}$ by

$$
\begin{aligned}
f_0(n) &= n + \tfrac{1}{2} & \qquad g_0(z) &= \lfloor \mathrm{Re}(z) \rfloor \\
f_1(n) &= in & g_1(z) &= \mathrm{Im}(z) \\
f_3(n) &= n + \tfrac{1}{2} & g_3(z) &= \lfloor z \rfloor .
\end{aligned}
$$

These are all continuous, because the floor function is continuous away from integer arguments. In each case we have $g_i f_i = \mathrm{id}$ and $f_i g_i$ is homotopic to the identity by a linear homotopy [2]. The spaces X_2 and X_4 have nontrivial H_1 and so cannot be homotopy equivalent to X_0, X_1 and X_3 [2]. The space X_2 is path-connected but X_4 is not, so X_2 is not homotopy equivalent to X_4 [1].

(iii) **Similar examples have been seen, but this is a bit harder than most of them.**
If we remove a point from X_0 we obtain a space with nontrivial H_1 but the same path components. However, if we remove a point from X_1 or X_3, we obtain a space with trivial H_1 and an extra path component. It follows that X_0 is not homeomorphic to X_1 or X_3 [2]. However, X_1 is a disjoint union of countably many copies of $\mathbb{R}$, and X_3 is a disjoint union of countably many copies of $(0,1)$, and $\mathbb{R}$ is homeomorphic to $(0,1)$, so X_1 is homeomorphic to X_3 [2]. Explicitly, we can define a homeomorphism $f\colon X_1 \to X_3$ by $f(x+ni) = n + \frac{1}{2} + x/(2\sqrt{1+x^2})$. As homeomorphism implies homotopy equivalence, part (ii) implies that there can be no further homeomorphisms. [2]

(2)

(a) What does it mean to say that a topological space X is *compact*? If your explanation relies on any auxiliary terms, then you should define them. **(3 marks)**

(b) Let X be compact topological space, and let Y be a closed subset of X.

(i) Define the subspace topology on Y. **(2 marks)**

(ii) Prove that when equipped with the subspace topology, Y is again compact. **(5 marks)**

(iii) Give an example of a compact space X and a compact subpace Y such that Y is not closed in X. **(3 marks)**

(iv) Explain a commonly-satisfied condition on X that guarantees that compact subspaces are closed. If your explanation relies on any auxiliary terms, then you should define them. However, you need not prove anything. **(3 marks)**

(c) Put $X = \mathbb{Z} \times \mathbb{Z}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid 100 < x^2 + y^2 < 10000\}$, considered as subspaces of the plane $\mathbb{R}^2$.

(i) Is X compact? **(1 marks)**

(ii) Is Y compact? **(1 marks)**

(iii) Is $X \cap Y$ compact? **(2 marks)**

Justify your answers.

(d) Let X be a metric space such that $X \setminus \{x\}$ is compact for all $x \in X$. Prove that X is finite. **(5 marks)**

Solution:

(a) **Bookwork** Let X be a topological space. By an *open cover* of X we mean a family $(U_i)_{i \in I}$ of open subsets of X, such that each point $x \in X$ lies in U_i for at least one index i [1]. A *finite subcover* of such a cover is a finite subset $J = \{j_1, \dots, j_n\} \subseteq I$ such that $(U_j)_{j \in J}$ is still a cover, or equivalently $X = U_{j_1} \cup \dots \cup U_{j_n}$ [1]. We say that X is *compact* if every open cover has a finite subcover [1].

(b) (i) **Bookwork** For the subspace topology on Y, we declare that a subset $V \subseteq Y$ is open iff there exists an open subset U of X such that $V = U \cap Y$ [2].

(ii) **Bookwork** Suppose that X is compact, and that Y is closed in X, which means that the set $U^* = X \setminus Y$ is open in X.

Let $(V_i)_{i \in I}$ be a family of subsets of Y that are open with respect to the subspace topology; we must show that this has a finite subcover [1]. As each V_i is open in the subspace topology, we can choose an open subset U_i of X such that $V_i = U_i \cap Y$ [1]. We find that the sets U_i together with U^* cover all of the compact space X [1], so there must be a finite subcover [1]. This means that there exists a finite subset $J \subseteq I$ such that $X = U^* \cup \bigcup_{j \in J} U_j$. In particular, for $y \in Y$ we note that y cannot lie in U^* so it must lie in one of the sets U_j with $j \in J$, but that means that $y \in Y \cap U_j = V_j$. This shows that $Y = \bigcup_{j \in J} V_j$ as required [1].

(iii) **Unseen** Take $X = \{0, 1\}$ with the indiscrete topology, and $Y = \{0\}$. Then Y is compact (as it is finite) but not closed. [3]

(iv) **Bookwork** A space X is said to be *Hausdorff* if for all $x, y \in X$ with $x \neq y$, there exist open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$ [1]. If X is Hausdorff, then any compact subset of X is closed [2].

(c) **Similar problems seen** We use the standard fact that a subset of $\mathbb{R}^2$ is compact iff it is bounded and closed.

(i) The set X is unbounded and thus not compact. [1]

(ii) The set Y is not closed, and thus is not compact. [1]

(iii) For $(x, y) \in X \cap Y$ we have $x, y \in \mathbb{Z}$ with $x^2 + y^2 < 10000$ so $x, y \in \{-99, -98, \dots, 98, 99\}$. This shows that $X \cap Y$ is finite and so is compact. [2]

(d) **Unseen** Let X be a metric space, so X is Hausdorff [1]. Suppose that for each $x \in X$, the set $X \setminus \{x\}$ is compact. As in (b)(iv) this means that $X \setminus \{x\}$ is closed, so $\{x\}$ is open in X [2]. If X is empty then it is certainly finite. Otherwise we can choose $a \in X$. By hypothesis the set $X \setminus \{a\}$ is compact, so the open cover by sets $\{x\}$ with $x \neq a$ must have a finite subscover [1]. This forces the set $X \setminus \{a\}$ to be finite, and it follows that X is finite as well [1].

(3) Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. **(3 marks)**

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. **(7 marks)**

(c) Explain how the connecting homomorphism $\delta\colon H_n(W) \to H_{n-1}(U)$ is defined in terms of snakes. If any further lemmas are needed to ensure that your definition is meaningful, then you should state those lemmas carefully, but you need not prove them. **(4 marks)**

(d) Consider the following example. For each $k \in \mathbb{Z}$ we have

$$\begin{aligned} U_k &= \mathbb{Z}/24 = \mathbb{Z}/(2^3 \times 3) & d^U(x) &= 12x = 2^2 \times 3 \times x \\ V_k &= \mathbb{Z}/1296 = \mathbb{Z}/(2^4 \times 3^4) & d^V(x) &= 36x = 2^2 \times 3^2 \times x \\ W_k &= \mathbb{Z}/54 = \mathbb{Z}/(2 \times 3^3) & d^W(x) &= -18x = -2 \times 3^2 \times x. \end{aligned}$$

The maps

$$U_k \xrightarrow{i} V_k \xrightarrow{p} W_k$$

are $i(a \pmod{24}) = 54a \pmod{1296}$ and $p(b \pmod{1296}) = b \pmod{54}$.

(i) Check that i and p are chain maps. (You may assume that they give a short exact sequence.) **(3 marks)**

(ii) Calculate the groups $H_k(U)$, $H_k(V)$ and $H_k(W)$. **(5 marks)**

(iii) By finding an appropriate snake, calculate the homomorphism $\delta\colon H_k(W) \to H_{k-1}(U)$. **(3 marks)**

Solution:

(a) **Bookwork** For each n, the map $i_n\colon U_n \to V_n$ is injective, the map $p_n\colon V_n \to W_n$ is surjective, and the image of i_n is the same as the kernel of p_n. [3]

(b) **Bookwork** Consider an element $c \in H_n(W)$. As $H_n(W) = Z_n(W)/B_n(W)$ by definition, we can certainly choose $w \in Z_n(W)$ such that $c = [w]$ [1]. As the sequence $U \xrightarrow{i} V \xrightarrow{p} W$ is short exact, we know that $p\colon V_n \to W_n$ is surjective, so we can choose $v \in V_n$ with $p(v) = w$ [1]. As p is a chain map we have $p(d(v)) = d(p(v)) = d(w) = 0$ (the last equation because $w \in Z_n(W)$) [1]. This means that $d(v) \in \ker(p)$, but $\ker(p) = \mathrm{img}(i)$ because the sequence is exact, so we have $u \in U_{n-1}$ with $i(u) = d(v)$ [1]. Note also that $i(d(u)) = d(i(u)) = d(d(v)) = 0$ (because i is a chain map and $d^2 = 0$) [1]. On the other hand, exactness means that i is injective, so the relation $i(d(u)) = 0$ implies that $d(u) = 0$ [1]. This shows that $u \in Z_{n-1}(U)$, so we can put $a = [u] \in H_{n-1}(U)$ [1]. We now have a snake (c, w, v, u, a) starting with c as required.

(c) **Bookwork** In addition to (b), we need the following lemma: given any two snakes (c, w, v, u, a) and (c, w', v', u', a') that both start with c, the endpoints a and a' are also the same [2]. This makes it possible to define $\delta\colon H_n(W) \to H_{n-1}(U)$ by the following rule: for any element $c \in H_n(W)$, we define $\delta(c)$ to be the endpoint of any snake that starts with c [2].

(d) **Similar examples seen**

(i) To show that i is a chain map, we must show that $d^V(i(x)) = i(d^U(x))$ in $\mathbb{Z}/1296$ for all $x \in \mathbb{Z}/24$, or equivalently that $54 \times 12 \times k = 36 \times 54 \times k \pmod{1296}$ for all $k \in \mathbb{Z}$. This holds because $(36\times54)-(54\times12) = 54 \times 24 = 2 \times 3^3 \times 2^3 \times 3 = 1296$ [2]. Similarly, to show that p is a chain map we just need to check that $36 = -18 \pmod{54}$, which is clear [1].

(ii) For $H_n(U)$ we note that $12k$ is divisible by 24 iff k is divisible by 2, so $Z_n(U) = \{0, 2, 4, \dots, 22\} \simeq \mathbb{Z}/12$, but $B_n(U) = \{0, 12\}$ so $H_n(U) \simeq \mathbb{Z}/6$, with generator $a = [2]$. [2]
For $H_n(V)$ we note that $36k$ is divisible by $1296 = 36^2$ iff k is divisible by 36, so $Z_n(V) = B_n(V) = 36V_n$ and $H_n(V) = 0$. [1]
For $H_n(W)$ we note that $-18k$ is divisible by $54 = 3\times18$ iff k is divisible by 3, so $Z_n(W) = \{0, 3, 6, \dots, 51\} \simeq \mathbb{Z}/18$. On the other hand, $B_n(W) = \{0, 18, 36\} \simeq \mathbb{Z}/3$, so $H_n(W) \simeq \mathbb{Z}/6$ with generator $c = [3]$. [2]

(iii) The sequence

$$(c, 3 \pmod{54}, 3 \pmod{1296}, 108 \pmod{1296}, 2 \pmod{24}, a)$$

is a snake, proving that $\delta(c) = a$. Thus, the homomorphism $\delta\colon (\mathbb{Z}/6).c \to (\mathbb{Z}/6).a$ is just given by $\delta(kc) = ka$. [3]

(4) For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A topological space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact. **(5 marks)**

(b) Subsets $A, B, C \subseteq \mathbb{R}^2$ such that A, B, C, $A \cup B$, $A \cup C$ and $B \cup C$ are all contractible, but $A \cup B \cup C$ is not contractible. **(5 marks)**

(c) A topological space X with two open subsets U and V such that U, V and $U \cap V$ are all homotopy equivalent to S^1, and $X = U \cup V$, and X is homotopy equivalent to S^4. **(5 marks)**

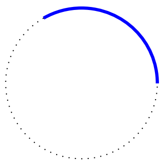
(d) A path connected space X such that $H_*(X)$ is not isomorphic to $H_*(X \times X)$. **(5 marks)**

(e) Spaces X and Y such that X is path connected, Y is not path connected, and $H_k(X) \simeq H_k(Y)$ for all k. **(5 marks)**

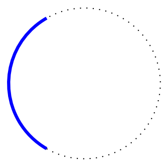
Solution:

(a) Take $X = S^1 \subset \mathbb{C}$ and $Y = X \setminus \{-1\}$ and $Z = X \setminus \{1\}$. Then neither Y nor Z is closed in $\mathbb{C}$, so they are both noncompact. However, $Y \cup Z = X$, and this is bounded and closed in $\mathbb{C}$ and is therefore compact. [5]

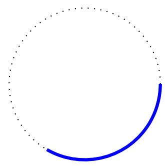
(b) Take A, B and C as follows:



$A = \{e^{2\pi i t/3} \mid 0 \leq t \leq 1\}$

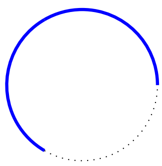


$B = \{e^{2\pi i t/3} \mid 1 \leq t \leq 2\}$

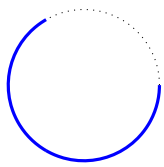


$C = \{e^{2\pi i t/3} \mid 2 \leq t \leq 3\}$

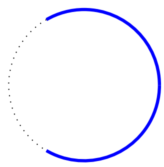
These are clearly contractible, as are the unions $A \cup B$, $B \cup C$ and $C \cup A$:



$A \cup B$



$B \cup C$



$C \cup A$

However, $A \cup B \cup C$ is the full circle S^1, which is not contractible. [5]

(c) This is not possible. If X, U and V were as specified, we would have $H_4(U) \simeq H_4(V) \simeq H_4(S^1) = 0$ and $H_3(U \cap V) \simeq H_3(S^1) \simeq 0$, whereas $H_4(X) \simeq H_4(S^4) \simeq \mathbb{Z}$. Thus, the Mayer-Vietoris sequence $H_4(U) \oplus H_4(V) \to H_4(X) \to H_3(U \cap V)$ would have the form $0 \to \mathbb{Z} \to 0$, which is not exact. [5]

(d) Take $X = S^1$, so $X \times X$ is a torus. It is clear that X is path connected, and standard calculations give $H_1(X) \simeq \mathbb{Z}$ and $H_1(X \times X) \simeq \mathbb{Z}^2$, so $H_*(X)$ is not isomorphic to $H_*(X \times X)$. [5]

(e) This is not possible. For any space Z we know that $H_0(Z)$ is the free abelian group generated by $\pi_0(Z)$, so $H_0(Z) \simeq \mathbb{Z}$ iff Z is path connected. Thus if X is path connected and Y is not, we cannot have $H_0(X) \simeq H_0(Y)$. [5]

(5) Consider S^1 as the unit circle in $\mathbb{R}^2$ as usual. Let X be a path connected space, and put

$$U = \{(t, x) \in S^1 \times X \mid t \neq (0, 1)\}$$
$$V = \{(t, x) \in S^1 \times X \mid t \neq (0, -1)\}.$$

We use the usual notation for inclusion maps:

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "k"] \\ V \arrow[r, "l"'] & S^1 \times X. \end{tikzcd}$$

(a) Define maps $f, g\colon X \to U \cap V$ such that f gives a homotopy equivalence from X to one path component of $U \cap V$, and g gives a homotopy equivalence from X to the other path component of $U \cap V$. **(4 marks)**

(b) Prove that the map $i' = i \circ f\colon X \to U$ is homotopic to $i \circ g$, and also that i' is a homotopy equivalence. (You can then assume without further argument that the map $j' = j \circ f\colon X \to V$ is homotopic to $j \circ g$, and that j' is a homotopy equivalence.) **(6 marks)**

(c) Deduce descriptions (in terms of $H_*(X)$) of the homology groups $H_p(U \cap V)$, $H_p(U)$ and $H_p(V)$, and the homomorphism

$$\alpha = \begin{bmatrix} i_* \\ -j_* \end{bmatrix} : H_p(U \cap V) \to H_p(U) \oplus H_p(V).$$

Find the kernel and image of α. **(8 marks)**

(d) Show that every element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), 0) + \alpha(b)$ for a unique pair $(a, b) \in H_p(X) \oplus H_p(X)$. **(3 marks)**

(e) Deduce that there is a short exact sequence $H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$. **(4 marks)**

Solution:

(a) The path components of $S^1 \setminus \{(0, 1), (0, -1)\}$ are $A = [(-1, 0)] = \{(x, y) \in S^1 \mid x < 0\}$ and $B = [(+1, 0)] = \{(x, y) \in S^1 \mid x > 0\}$, so the path components of $U \cap V$ are $A \times X$ and $B \times X$ [2]. Here A is contractible and contains $(-1, 0)$ so the map $f(x) = ((-1, 0), x)$ gives a homotopy equivalence from X to $A \times X$. Similarly, the map $g(x) = ((1, 0), x)$ gives a homotopy equivalence from X to $B \times X$ [2].

(b) We can define $h(t, x) = ((-\cos(\pi t), -\sin(\pi t)), x)$ for $0 \leq t \leq 1$. As $(-\cos(\pi t), -\sin(\pi t))$ lies on the bottom half of S^1, this does not pass through $(0, 1) \times X$ and so gives a continuous map $[0, 1] \times X \to U$. It satisfies $h(0, x) = ((-1, 0), x) = i(f(x)) = i'(x)$ and $h(1, x) = ((1, 0), x) = i(g(x))$, so this gives a homotopy between i' and $i \circ g$ [3]. We can also define $r\colon U \to X$ by $r(t, x) = x$. Then $r \circ i' = \text{id}$, and contractibility of $S^1 \setminus \{(0, 1)\}$ ensures that $i'r$ is homotopic to the identity [3].

(c) As $f\colon X \to A \times X$ and $g\colon X \to B \times X$ are homotopy equivalences, we see that every element of $H_p(U \cap V)$ can be written as $f_*(a) + g_*(b)$ for a unique pair $(a, b) \in H_p(X) \oplus H_p(X)$. [2] Similarly, any element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), j'_*(b))$ for a unique pair $(a, b) \in H_p(X) \oplus H_p(X)$[2]. As $i_* f_* = i_* g_* = i'_*$ and $j_* f_* = j_* g_* = j'_*$ we see that

$$\alpha(f_*(a) + g_*(b)) = (i'_*(a + b),\ -j'_*(a + b)).\text{[2]}$$

This means that

$$\ker(\alpha) = \{f_*(a) - g_*(a) \mid a \in H_p(X)\} \simeq H_p(X)\text{[1]}$$
$$\operatorname{img}(\alpha) = \{(i'_*(c), -j'_*(c)) \mid c \in H_p(X)\} \simeq H_p(X)\text{[1]}.$$

(d) We now see that every element $(i'_*(a), j'_*(b)) \in H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a+b), 0) + (i'_*(-b), -j'_*(-b))$ with the second term lying in $\operatorname{img}(\alpha)$, and this decomposition is unique [3].

(e) From the exact sequence

$$H_p(U \cap V) \xrightarrow{\alpha} H_p(U) \oplus H_p(V) \to H_p(S^1 \times X) \xrightarrow{\delta} H_{p-1}(U \cap V) \xrightarrow{\alpha} H_{p-1}(U) \oplus H_{p-1}(V)$$

we get a short exact sequence

$$(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \to H_p(S^1 \times X) \to \ker(\alpha_{p-1})\text{[2]}$$

Part (d) gives an isomorphism $(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \simeq H_p(X)$ [1]. Part (c) gives an isomorphism $\ker(\alpha_{p-1}) \simeq H_{p-1}(X)$ [1]. We therefore have a short exact sequence

$$H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$$

as claimed.