MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 11

Please hand in Exercises 2 and 4 by the Wednesday lecture of Week 5. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. Give an elementary proof (just using real analysis, not algebraic topology) of the case $n = 1$ of the Brouwer Fixed Point Theorem.

Exercise 2. Consider B^2 as a subset of $\mathbb{C}$, so $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Check that the following formulae define continuous maps $f_k \colon B^2 \to B^2$, and find their fixed points.

$$f_1(z) = -z \qquad f_2(z) = \overline{z} \qquad f_3(z) = \frac{2z-1}{2-z} \qquad f_4(z) = |z|z + 1 - |z|$$

(Note that in particular, you need to show that $f_i(z) \in B^2$ whenever $z \in B^2$. For $f_3(z)$, you can give a direct argument or you can recall some relevant theory from Complex Analysis. To understand $f_4(z)$, think about the same expression with $|z|$ replaced by an arbitrary real number t.)

Exercise 3. Suppose that $n > 0$. For each of the spaces $X = S^n, \mathbb{R}^n, OB^n$ define a continuous map $f \colon X \to X$ that has no fixed points.

Exercise 4. You can assume all homology calculations mentioned in the notes. Show that

(a) Neither of $\mathbb{R}P^1$ and $\mathbb{R}P^2$ is a homotopy retract of the other.
(b) The torus T^2 is a homotopy retract of T^3, but T^3 is not a homotopy retract of T^2.
(c) S^1 is a retract of $S^3 \setminus S^1$
(d) OB^2 is a homotopy retract of B^2, but not an actual retract.

Exercise 5. Let $p, q \colon \mathbb{C} \to \mathbb{C}$ be continuous maps such that p is a polynomial of degree $n > 0$ and q satisfies $|q(x)| < 1$ for all $x \in \mathbb{C}$. By adapting the proof of the Fundamental Theorem of Algebra, prove that there exists $x \in \mathbb{C}$ such that $p(x) = q(x)$.