MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 9 — Solutions

Please hand in Exercises 2 and 9 by the Wednesday lecture of Week 3. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. Write down all isomorphism classes of abelian groups of order 2, 4 and 8. Write down all isomorphism classes of abelian groups of order 6, 10, 15.

Solution: Any finite abelian group can be expressed as a direct sum of terms $\mathbb{Z}/p^k$ with p prime and $k > 0$. From this we obtain the following lists:

- Order 2: $\mathbb{Z}/2$
- Order 4: $\mathbb{Z}/4$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.
- Order 8: $\mathbb{Z}/8$, $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Next, if p and q are distinct primes, the only way to make an abelian group of order pq is $\mathbb{Z}/p \oplus \mathbb{Z}/q$. (You might ask about $\mathbb{Z}/pq$, but that is isomorphic to $\mathbb{Z}/p \oplus \mathbb{Z}/q$, by the Chinese Remainder Theorem.) Thus, for orders 6, 10 and 15 we just have $\mathbb{Z}/2 \oplus \mathbb{Z}/3$, $\mathbb{Z}/2 \oplus \mathbb{Z}/5$ and $\mathbb{Z}/3 \oplus \mathbb{Z}/5$.

Exercise 2.

(a) If there is an exact sequence

$$0 \to \mathbb{Z}/4 \xrightarrow{\alpha} A \xrightarrow{\beta} \mathbb{Z}/2 \to 0,$$

what are the possible isomorphism types for A? If you think that A could be $\mathbb{Z}/10$, for example, you should give explicit maps $\mathbb{Z}/4 \xrightarrow{\alpha} \mathbb{Z}/10 \xrightarrow{\beta} \mathbb{Z}/2$ and check that they are well-defined and give a short exact sequence.

Optional extra: If there is an exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\alpha} \mathbb{Z}/4 \xrightarrow{\beta} B \xrightarrow{\gamma} \mathbb{Z}/4 \oplus \mathbb{Z}/2 \xrightarrow{\delta} C \xrightarrow{\epsilon} \mathbb{Z}/2 \to 0,$$

what are the possible isomorphism types for B and C? (There are many possibilities.)

(b) Show that if there is an exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} \mathbb{Z} \to 0$ then $B \cong A \oplus \mathbb{Z}$. You should start by showing that there is a homomorphism $\sigma \colon \mathbb{Z} \to B$ such that $\beta\sigma = 1$.

Solution:

(a) Given a short exact sequence $U \to V \to W$ of finite abelian groups, we always have $|V| = |U|.|W|$. Thus, in this problem we have $|A| = 8$, so A is isomorphic to $A_0 = \mathbb{Z}/8$ or $A_1 = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ or $A_2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. However, every element $x \in A_2$ has $2x = 0$, so there cannot be an injective homomorphism $\alpha \colon \mathbb{Z}/4 \to A_2$, so A cannot be isomorphic to A_2. This just leaves A_0 and A_1. Both of these cases can occur, because there are short exact sequences

$$\mathbb{Z}/4 \xrightarrow{\alpha_0} \mathbb{Z}/8 \xrightarrow{\beta_0} \mathbb{Z}/2 \qquad\qquad \mathbb{Z}/4 \xrightarrow{\alpha_0} \mathbb{Z}/4 \oplus \mathbb{Z}/2 \xrightarrow{\beta_0} \mathbb{Z}/2$$

given by

$$\alpha_0(k \pmod 4) = 2k \pmod 8 \qquad\qquad \beta_0(k \pmod 8) = k \pmod 2$$
$$\alpha_1(k \pmod 4) = (k \pmod 4, 0) \qquad\qquad \beta_1(k \pmod 4, m \pmod 2) = m \pmod 2.$$

We now consider the second exact sequence. We'll put $U = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ for brevity, so $|U| = 8$. Let P, Q and R be the images of β, γ and δ, or equivalently the kernels of γ, δ and ϵ. The exact sequence can then be separated into short exact sequences as follows:

$$\mathbb{Z}/2 \to \mathbb{Z}/4 \to P \qquad P \to B \to Q \qquad Q \to U \to R \qquad R \to C \to \mathbb{Z}/2.$$

It is easy to see that the first of these forces P to be $\mathbb{Z}/2$. From the other short exact sequences we obtain

$$|B| = |P||Q| = 2|Q| \qquad 8 = |U| = |Q||R| \qquad |C| = 2|R|.$$

From the middle equation we see that the pair $(|Q|, |R|)$ is either $(1, 8)$ or $(2, 4)$ or $(4, 2)$ or $(8, 1)$. In the $(1, 8)$ case we have $Q = 0$ so the first two short exact sequences give $B = P = \mathbb{Z}/2$ and $R = U = \mathbb{Z}/4 \oplus \mathbb{Z}/2$, so the last short exact sequence looks like $\mathbb{Z}/4 \oplus \mathbb{Z}/2 \to C \to \mathbb{Z}/2$. From this one can check that C is isomorphic to one of the groups $V_0 = \mathbb{Z}/8 \oplus \mathbb{Z}/2$ or $V_1 = \mathbb{Z}/4 \oplus \mathbb{Z}/4$ or $V_2 = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The $(8, 1)$ case is similar, and we find that $C = \mathbb{Z}/2$ and B is V_0, V_1 or V_2.

Now suppose instead that $(|Q|,|R|)=(2,4)$, so $(|B|,|C|)=(4,8)$. In this case it turns out that B can be either of the groups $\mathbb{Z}/4$ or $\mathbb{Z}/2\oplus\mathbb{Z}/2$ of order 4, and C can be any of the groups $\mathbb{Z}/8$, $\mathbb{Z}/4\oplus\mathbb{Z}/2$ or $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ of order 8, and there are no further constraints. The situation is similar if $(|Q|,|R|)=(4,2)$.

(b) Suppose we have an exact sequence

$$0\to A\xrightarrow{\alpha}B\xrightarrow{\beta}\mathbb{Z}\to 0.$$

Exactness means in particular that β is surjective, so we can choose $b_0\in B$ with $\beta(b_0)=1$. We can then define $\phi\colon A\oplus\mathbb{Z}\to B$ by $\phi(a,n)=\alpha(a)+n\,b_0$.

We claim that ϕ is an isomorphism, or equivalently, that it is both injective and surjective.

To prove that ϕ is injective, suppose that we have (a,n) with $\phi(a,n)=0$; it will be enough to show that $(a,n)=(0,0)$. The equation $\phi(a,n)=0$ means that $\alpha(a)+n\,b_0=0$. Applying β gives $\beta\alpha(a)+n\,\beta(b_0)=0$. Exactness implies that $\beta\alpha=0$, and we know that $\beta(b_0)=1$, so we get $n=0$. Using this, the equation $\phi(a,n)=0$ becomes $\alpha(a)=0$, but exactness also implies that α is injective, so we must have $a=0$, as required.

To prove that ϕ is surjective, consider an arbitrary element $b\in B$. Put $n=\beta(b)\in\mathbb{Z}$, and $b'=b-n\,b_0$. We have $\beta(b')=\beta(b)-n\beta(b_0)=n-n.1=0$, so $b'\in\ker(\beta)$. However, exactness means that $\ker(\beta)=\operatorname{img}(\alpha)$, so we can find $a\in A$ with $b'=\alpha(a)$. As $b'=b-n\,b_0$, this can be rearranged to give $b=\alpha(a)+n\,b_0=\phi(a,n)$. This proves that b lies in the image of ϕ, as required.

Exercise 3.

(a) Let $\phi\colon A\to B$ be a homomorphism between Abelian groups. Show that $\phi(na)=n\phi(a)$ for all $a\in A$ and $n\in\mathbb{Z}$. (Start with the case $n\geq 0$ and use induction.)

(b) Let B be an Abelian group, and let $\phi\colon\mathbb{Z}^2\to B$ be a homomorphism. Show that there are elements $u,v\in B$ such that $\phi(n,m)=nu+mv$ for all $(n,m)\in\mathbb{Z}^2$.

(c) List all the homomorphisms from $\mathbb{Z}^2$ to $\mathbb{Z}/9$. How many of them are surjective?

(d) Prove that there is no homomorphism $\phi\colon\mathbb{Z}/4\to\mathbb{Z}/12$ such that $\phi(1)=1$.

(e) How much can you say about homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$ for arbitrary natural numbers n,m?

Solution:

(a) Let $\phi\colon A\to B$ be a homomorphism between Abelian groups, so

$$\phi(a+a')=\phi(a)+\phi(a')$$

for all $a,a'\in A$; we need to show that $\phi(na)=n\phi(a)$ for all $a\in\mathbb{Z}$. The element na is effectively defined by recursion: we have $0.a=0$ and $1.a=a$ and $na=a+(n-1)a$ for $n>1$. For negative n, we use the above procedure to define $|n|a$ and then we put $na=-(|n|a)$.

By putting $a=a'=0$ in the displayed equation, we see that $\phi(0)=\phi(0)+\phi(0)$; after subtracting $\phi(0)$ from both sides, we conclude that $\phi(0)=0$, which is the case $n=0$ of the claim. The case $n=1$ is obvious. For $n>1$ we may assume inductively that $\phi((n-1)a)=(n-1)\phi(a)$. By putting $a'=(n-1)a$ in the displayed equation, we find that $\phi(a+(n-1)a)=\phi(a)+(n-1)\phi(a)$, or equivalently $\phi(na)=n\phi(a)$. This proves the claim for all $n\geq 0$.

Next, put $a=c$ and $a'=-c$ in the displayed equation to get $0=\phi(0)=\phi(c)+\phi(-c)$, which implies that $\phi(-c)=-\phi(c)$ for all $c\in A$. Now, if $n<0$ we can put $c=|n|a$ to find that

$$\phi(na)=\phi(-c)=-\phi(c)=-\phi(|n|a)=-|n|\phi(a)=n\phi(a),$$

as claimed.

(b) Let $\phi\colon\mathbb{Z}^2\to B$ be a homomorphism of Abelian groups. Put $u=\phi(1,0)\in B$ and $v=\phi(0,1)\in B$. For any $n,m\in\mathbb{Z}$ we then have $(n,m)=(n,0)+(0,m)=n(1,0)+m(0,1)$, so

$$\phi(n,m)=n\phi(1,0)+m\phi(0,1)=nu+mv,$$

as required.

(c) The group $B:=\mathbb{Z}/9$ consists of the elements $(u \bmod 9)$ for $u=0,1,\dots,8$. Thus for any pair of integers $u,v\in\{0,\dots,8\}$ we have a homomorphism $\phi_{uv}\colon\mathbb{Z}^2\to B$ defined by

$$\phi_{uv}(n,m)=(nu+mv \bmod 9).$$

This gives $9^2=81$ different homomorphisms, and part (b) tells us that this is a complete list.

Suppose that u is not divisible by 3; I claim that there is an integer n such that $nu=1\pmod 9$. Indeed, as the only factors of 9 are 1, 3 and $3^2=9$, we see that u and 9 have no common divisors other than 1, so by the theory of gcd's we see that $un+9m=1$ for some integers n,m, or in other words $nu=1-9m=1\pmod 9$ as claimed. We now see that $\phi uv(kn,0)=(kun \bmod 9)=(k \bmod 9)$ for all $k\in\mathbb{Z}$, and this implies that ϕ_{uv} is surjective. Similarly, if v is not divisible by 3 then ϕ_{uv} is surjective.

Now define

$$A = \{(3m \bmod 9) \mid n \in \mathbb{Z}\} = \{(0 \bmod 9), (3 \bmod 9), (6 \bmod 9)\} \subset \mathbb{Z}/9;$$

this is easily seen to be a subgroup of B, with precisely three elements. Suppose that u and v are divisible by 3, say $u = 3u'$ and $v = 3v'$. Then $\phi_{uv}(n, m) = (3(nu' + mv') \bmod 9) \in A$. This shows that only elements in A can be hit by ϕ_{uv}, so in particular the element $(1 \bmod 9)$ is not in the image of ϕ_{uv}, so ϕ_{uv} is not surjective.

The conclusion is that ϕ_{uv} is surjective iff at least one of u and v is not divisible by 3. Looking at this the other way around, we see that ϕ_{uv} is not surjective iff u and v both lie in the set $\{0, 3, 6\}$, which gives $3^2 = 9$ possibilities for the pair (u, v). This leaves $81 - 9 = 72$ pairs (u, v) for which ϕ_{uv} is surjective.

(d) Let $\phi\colon \mathbb{Z}/4 \to \mathbb{Z}/12$ be a homomorphism. Put $a = (1 \bmod 4) \in \mathbb{Z}/4$, so $4a = (4 \bmod 4) = 0$. Put $b = \phi(a) \in \mathbb{Z}/12$; then $4b = \phi(4a) = \phi(0) = 0$. However, $4(1 \bmod 12) = (4 \bmod 12) \neq 0$, so (as claimed) we cannot have $b = (1 \bmod 12)$.

(e) Let n and m be natural numbers with greatest common divisor d. We then have $n = pd$ and $m = qd$ for certain coprime numbers p, q, and we can choose integers x, y such that $px + qy = 1$.

For each $u \in \{0, 1, \dots, d-1\}$ we can define $\phi_u\colon \mathbb{Z} \to \mathbb{Z}/m$ by $\phi_u(k) = (kqu \bmod m)$. We then have

$$\phi_u(ni) = (niqu \bmod m) = (pdiqu \bmod m) = (pium \bmod m) = 0$$

for all $i \in \mathbb{Z}$, so $\phi_u(n\mathbb{Z}) = \{0\}$. This gives a well-defined map $\overline{\phi}_u\colon \mathbb{Z}/n \to \mathbb{Z}/m$ with

$$\overline{\phi}_u(k \bmod n) = \phi_u(k) = (kqu \bmod m)$$

for all k.

Note that $\overline{\phi}_u(1 \bmod n) = (qu \bmod m)$. The numbers $(0, q, \dots, q(d-1))$ are all different and they all lie in the range from 0 to $m - 1 = qd - 1$, so no two of them are congruent modulo m. It follows that the maps $\overline{\phi}_0, \dots, \overline{\phi}_{d-1}$ are all different.

Finally, I claim that this is a complete list of *all* the homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$. To see this, let $\psi\colon \mathbb{Z}/n \to \mathbb{Z}/m$ be a homomorphism. We then have $\psi(1 \bmod n) = (v \bmod m)$ for some $v \in \{0, \dots, m-1\}$. It follows that

$$(nv \bmod m) = n\psi(1 \bmod n) = \psi(n \bmod n) = \psi(0) = 0,$$

so nv is divisible by m. We thus have $nv = mw$ for some w, or equivalently $pvd = qwd$ or $pv = qw$. Put $u = wx + yv$; we find that $qu = qwx + qyv = pvx + qyv = (px + qy)v = v$. Thus $\psi(1 \bmod n) = (qu \bmod m)$, and we deduce that $\psi(k \bmod n) = (kqu \bmod m)$ for all k, so $\psi = \overline{\phi}_u$. We have thus found all the homomorphisms, as claimed.

Exercise 4. What is the minimum number of generators for $\mathbb{Z}/2 \oplus \mathbb{Z}/2$? What is the minimum number of generators for $\mathbb{Z}/2 \oplus \mathbb{Z}/3$? Is $\mathbb{Q}$ a finitely generated abelian group?

Solution: The group $A = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ has elements 0 and $a = (1, 0)$ and $b = (0, 1)$ and $c = (1, 1)$, which satisfy $2a = 2b = 2c = 0$. As $2a = 0$ we see that the subgroup generated by a is just $\{0, a\}$, so in particular it is not the whole group. Similarly, the subgroup generated by b is $\{0, b\}$, and the subgroup generated by c is just $\{0, c\}$, so no single element generates the whole group. On the other hand, as $c = a + b$ we see that a and b together generate the group. Thus, the minimum possible number of generators is 2.

Now consider instead the group $B = \mathbb{Z}/2 \oplus \mathbb{Z}/3$. This is certainly generated by the elements $a = (1, 0)$ and $b = (0, 1)$, so two generators is enough. However, we actually only need one generator. Indeed, as 2 and 3 are coprime, the Chinese Remainder Theorem tells us that $B \simeq \mathbb{Z}/6$, and $\mathbb{Z}/6$ only needs one generator. Explicitly, if we take $c = (1, 1)$, and remember to work mod 2 in the first factor and mod 3 in the second factor, we get:

$$0c = (0,0) \qquad 1c = (1,1) \qquad 2c = (0,2)$$
$$3c = (1,0) \qquad 4c = (0,1) \qquad 5c = (1,2).$$

From this we see that $B = \{kc \mid 0 \leq k < 6\}$, so c generates B.

Finally, we claim that $\mathbb{Q}$ is not a finitely generated abelian group. Indeed, suppose we have a finite list of rational numbers $q_1, \dots, q_n$, and we let U denote the group that they generate; we must show that this is not all of $\mathbb{Q}$. We can write q_i as a_i/b_i for some integers a_i, b_i with $b_i > 0$. Put $b = b_1 b_2 \cdots b_n$. We then see that bq_i is an integer for all i. If $u \in U$ then $u = m_1 q_1 + \cdots + m_n q_n$ for some integers m_i, so $bu = \sum_i m_i(bq_i)$, so $bu \in \mathbb{Z}$. On the other hand, we can choose a prime number p that does not divide b, and then we find that $b.p^{-1} \notin \mathbb{Z}$, so $p^{-1} \notin U$. This shows that U is not all of $\mathbb{Q}$, as required.

Exercise 5. Let p and q be coprime integers, and let $\phi\colon \mathbb{Z} \to \mathbb{Z}/p$ be the homomorphism $\phi(n) = (nq \bmod p)$. Prove that ϕ is surjective. Prove also that the only homomorphism from $\mathbb{Z}/q$ to $\mathbb{Z}/p$ is the zero homomorphism

Solution: Let p and q be coprime, so $px+qy=1$ for some integers x,y. Define $\phi\colon \mathbb{Z}\to\mathbb{Z}/p$ by $\phi(n)=(nq \bmod p)$. For any element $(k \bmod p)\in\mathbb{Z}/p$, we have $\phi(ky)=(kyq \bmod p)$ but $yq=1-px=1 \pmod{p}$ so $kyq=k \pmod{p}$, so $(k \bmod p)=\phi(ky)$. This proves that ϕ is surjective.

Now let $\psi\colon \mathbb{Z}/q\to\mathbb{Z}/p$ be a homomorphism. The equation $1=px+qy$ implies that $1=qy \pmod{p}$ so $k=qky \pmod{p}$ for all k, so $\psi(k \bmod p)=\psi(qky \bmod p)=q\psi(ky \bmod p)$. However, $\psi(ky \bmod p)\in\mathbb{Z}/q$ and $qb=0$ for all $b\in\mathbb{Z}/q$ so $\psi(k \bmod p)=0$. This holds for all k, so $\psi=0$ as claimed.

Exercise 6. Let A be a finite Abelian group, and let B be a free Abelian group. Prove that if $\phi\colon A\to B$ is a homomorphism, then $\phi=0$.

Solution: Let A be a finite Abelian group (of order n say), and let B be a free Abelian group (so $B=\mathbb{Z}[D]$ for some set D).

Let u be a nonzero element of B; I claim that nu is also nonzero. To see this, write u in the form $n_1[d_1]+\ldots+n_r[d_r]$ for some elements $d_1,\ldots,d_r\in D$ and some integers $n_1,\ldots,n_r\in\mathbb{Z}$. If any two of the elements d_i are actually the same, then we can collect the corresponding terms together (e.g. $2[d]+3[d']+4[d]=6[d]+3[d']$), and if any coefficient n_i is zero then we can omit the corresponding term. After performing these processes as many times as possible, we get an expression $u=m_1[e_1]+\ldots+m_s[e_s]$ where $m_i\neq 0$ for all i and the elements $e_1,\ldots,e_s\in D$ are all distinct. (If we allowed the possibility $u=0$ then we might end up with the case $s=0$ with no terms at all on the right hand side, but we are assuming that $u\neq 0$ so we must have $s\geq 1$). We now have $nu=nm_1[e_1]+\ldots+nm_s[e_s]$. Each coefficient nm_i is nonzero and the e_i's are all distinct, so no cancellation can occur. It follows that $nu\neq 0$, as claimed.

Now let $\phi\colon A\to B$ be a homomorphism. If $a\in A$ then Lagrange's theorem tells us that the order of a divides $n=|A|$, so $na=0$. We thus have $n\phi(a)=\phi(na)=\phi(0)=0$. By the previous paragraph, this can only happen if $\phi(a)=0$. Thus $\phi(a)=0$ for all a, in other words $\phi=0$.

Exercise 7. Suppose we have two sets D and E each with precisely two elements, say $D=\{p,q\}$ and $E=\{r,s\}$. Define a function $\psi\colon D\to\mathbb{Z}[E]$ by

$$\psi(p)=3[r]+[s] \qquad\qquad \psi(q)=5[r]+2[s]$$

and let $\phi\colon\mathbb{Z}[D]\to\mathbb{Z}[E]$ be the linear extension of ψ. What is $\phi([p]-[q])$? What is $\phi(n[p]+m[q])$?

Now define a map $\zeta\colon E\to\mathbb{Z}[D]$ by

$$\zeta(r)=2[p]-[q] \qquad\qquad \zeta(s)=-5[p]+3[q]$$

and let ξ be the linear extension of ζ. What is $\xi\phi([p])$? Extend this calculation to show that $\xi=\phi^{-1}$.

Solution: First, we have

$$\phi([p]-[q])=\psi(p)-\psi(q)=3[r]+[s]-5[r]-2[s]=-2[r]-[s].$$

More generally, we have

$$\phi(n[p]+m[q])=n\psi(p)+m\psi(q)=3n[r]+n[s]+5m[r]+2m[s]=(3n+5m)[r]+(n+2m)[s].$$

Next, we have

$$\begin{aligned}\xi\phi[p]&=\xi(3[r]+[s])\\&=3\zeta(r)+\zeta(s)\\&=6[p]-3[q]-5[p]+3[q]=[p].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\xi\phi[q]&=\xi(5[r]+2[s])\\&=5\zeta(r)+2\zeta(s)\\&=10[p]-5[q]-10[p]+6[q]=[q].\end{aligned}$$

As $\xi\phi$ is a homomorphism $\mathbb{Z}[D]\to\mathbb{Z}[D]$, we deduce that

$$\xi\phi(n[p]+m[q])=n\xi\phi[p]+m\xi\phi[q]=n[p]+m[q].$$

As every element $u\in\mathbb{Z}[D]$ has the form $n[p]+m[q]$ for some n,m, we deduce that $\xi\phi(u)=u$ for all u, so $\xi\phi$ is the identity map.

In the other direction, we have

$$\begin{aligned}\phi\xi[r]&=2\phi[p]-\phi[q]=6[r]+2[s]-5[r]-2[s]=[r]\\ \phi\xi[s]&=-5\phi[p]+3\phi[q]=-15[r]-5[s]+15[r]+6[s]=[s].\end{aligned}$$

We deduce in the same way that $\phi\xi\colon\mathbb{Z}[E]\to\mathbb{Z}[E]$ is the identity map, so $\xi=\phi^{-1}$.

Exercise 8. Let $\phi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}/12$ be the homomorphism defined by

$$\phi(n,m) = (3n, (2n + 4m \bmod 12)).$$

Give an isomorphism $\psi\colon \mathbb{Z} \to \ker(\phi)$.

Solution: Define $\psi\colon \mathbb{Z} \to \mathbb{Z}^2$ by $\psi(k) = (0, 3k)$. We have $\phi\psi(k) = \phi(0, 3k) = (0, (12k \bmod 12)) = (0, 0)$, so $\psi(k) \in \ker(\phi)$ for all k. We can thus regard ψ as a homomorphism $\mathbb{Z} \to \ker(\phi)$. It is clearly injective (ie if $j \neq k$ then $(0, 3j) \neq (0, 3k)$). Now suppose $(n, m) \in \ker(\phi)$, so $(3n, (2n + 4m \bmod 12)) = (0, 0) \in \mathbb{Z}\times\mathbb{Z}/12$. This means that $3n = 0 \in \mathbb{Z}$ and $2n + 4m$ is divisible by 12. The first equation implies that $n = 0$, so $4m$ is divisible by 12, say $4m = 12k$ for some k. This gives $m = 3k$ so $(n, m) = (0, 3k) = \psi(k)$, proving that our homorphism $\psi\colon \mathbb{Z} \to \ker(\phi)$ is surjective. It is thus an isomorphism, as claimed.

Exercise 9. Consider the following sequences of abelian groups and homomorphisms. The degrees are indicated by the top row, so for example $B_0 = \mathbb{Z}/6$ and $B_1 = B_2 = \mathbb{Z}/4$.

For each sequence, decide whether it is a chain complex. If it is a chain complex, find the homology, and decide whether the sequence is exact.

$$\begin{array}{lccccccccccccccc}
\text{degree} & & & -2 & & -1 & & 0 & & 1 & & 2 & & 3 & & \\
A_* = & \cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{6} & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
B_* = & \cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}/6 & \xleftarrow{3} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \longleftarrow & 0 & \longleftarrow & \cdots \\
C_* = & \cdots & \longleftarrow & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow & \cdots \\
D_* = & \cdots & \longleftarrow & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \longleftarrow & \cdots \\
E_* = & \cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow & \cdots \\
F_* = & \cdots & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \longleftarrow & \cdots
\end{array}$$

(Here notation like $\mathbb{Z}/n \xrightarrow{p} \mathbb{Z}/m$ refers to the map $f\colon \mathbb{Z}/n \to \mathbb{Z}/m$ given by $f(i \pmod n) = pi \pmod m$. You should think about the conditions on n, m and p that are needed to make this well-defined.)

Solution: First, the map $\mathbb{Z}/n \xrightarrow{p} \mathbb{Z}/m$ is well-defined if and only if pn is divisible by m.

(a) Here d_1 is the map $\mathbb{Z} \xrightarrow{6} \mathbb{Z}$, and all other differentials are zero. As there is only one nonzero differential, it is clear that the composite of any two differentials is zero, so we have a chain complex. For $i \neq 0, 1$ we have $A_i = 0$ so $H_iA = 0$. The map d_1 is injective, so $Z_1A = 0$ so $H_1A = 0$. However, we have $Z_0 = \ker(0\colon \mathbb{Z} \to 0) = \mathbb{Z}$ and $B_0 = \operatorname{img}(6\colon \mathbb{Z} \to \mathbb{Z}) = 6\mathbb{Z}$ so $H_0A = \mathbb{Z}/6$. As $H_*A \neq 0$, this is not an exact sequence.

(b) The only composite that could possibly be nonzero is $d_1d_2\colon \mathbb{Z}/4 \to \mathbb{Z}/6$. This is multiplication by $3 \times 2 = 6$, and so is zero in $\mathbb{Z}/6$. Thus, we have a chain complex. The cycles and boundaries are as follows:

$$\begin{array}{lll}
Z_0B = \{0,1,2,3,4,5\} & Z_1B = \{0,2\} & Z_2B = \{0,2\} \\
B_0B = \{0,3\} & B_1B = \{0,2\} & B_2B = \{0\}.
\end{array}$$

Taking the quotients gives $H_0B = \mathbb{Z}/3$ and $H_1B = 0$ and $H_2B = \mathbb{Z}/2$. In particular, the homology is nontrivial so the sequence is not exact.

(c) Here the composite of any two adjacent maps is $\mathbb{Z} \xrightarrow{4} \mathbb{Z}$, which is nonzero. Thus, the sequence is not a chain complex, and the homology is not defined.

(d) Here the image of every map is $\{0, 2\}$, and this is the same as the kernel of the next map. Thus, we have a chain complex whose homology is zero, or in other words, an exact sequence.

(e) Here, whenever we have two adjacent maps, one of them is zero, so the composite is zero. This means that we have a chain complex. The cycles and boundaries are as follows:

$$\begin{array}{lllll}
Z_0E = \mathbb{Z} & Z_1E = 0 & Z_2E = \mathbb{Z} & Z_3E = 0 & Z_4E = \mathbb{Z}\cdots \\
B_0E = 2\mathbb{Z} & B_1E = 0 & B_2E = 2\mathbb{Z} & B_3E = 0 & B_4E = 2\mathbb{Z}\cdots
\end{array}$$

This gives $H_{2k}E = \mathbb{Z}/2$ (for $k \geq 0$) and $H_{2k+1}E = 0$ (for all k). It is also clear that $H_{2k}E = 0$ when $k < 0$.

(f) This is the same as (e) except that we have an additional homology group of $H_{-1}F = \mathbb{Z}$.