MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 10 — Solutions

Please hand in Exercises 2 and 3 by the Wednesday lecture of Week 4. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. Let U be an abelian group. Consider the chain complex

$$A_* = (U \xleftarrow{0} U \xleftarrow{1} U \xleftarrow{0} U \xleftarrow{1} U \xleftarrow{\cdots})$$

(with the first group in degree zero).

(a) What is H_*A?

(b) Define $f\colon A_* \to A_*$ by $f_0 = 1$ and $f_k = 0$ for all $k \neq 0$. Prove that f is chain-homotopic to the identity.

Solution:

(a) We have $Z_0A = U$ and $B_0A = 0$ so $H_0A = U$. For even $k > 0$ we have $Z_kA = 0 = B_kA$ so $H_kA = 0$, and the same applies for $k < 0$. For odd $k > 0$ we have $Z_kA = U = B_kA$ so $H_kA = U/U = 0$ again. In summary, we have $H_0A = U$ and all other homology groups are zero.

(b) Define maps $s_k\colon A_k \to A_{k+1}$ as follows:

$$U \xrightarrow{0} U \xrightarrow{1} U \xrightarrow{0} U \xrightarrow{1} U \to \cdots$$

On A_0 all relevant maps d and s are zero so $ds + ds = 0 = 1 - f_0$. If $k > 0$ is even then the maps $A_k \xrightarrow{d} A_{k-1} \xrightarrow{s} A_k$ are both equal to the identity on U, whereas the maps $A_k \xrightarrow{s} A_{k+1} \xrightarrow{d} A_k$ are both zero, so $ds + ds = 1$. If $k > 0$ is odd then the maps $A_k \xrightarrow{d} A_{k-1} \xrightarrow{s} A_k$ are both zero, whereas the maps $A_k \xrightarrow{s} A_{k+1} \xrightarrow{d} A_k$ are both equal to the identity on U so $ds + ds = 1$ again. As $f_k = 0$ in both these cases we can say that $ds + ds = 1 - f$ in all degrees.

One can check that it also works to take $s_k = 1\colon U_k \to U_{k+1}$ for all $k \geq 0$.

Exercise 2. Consider the chain complex U_* where $U_n = \mathbb{Z}/100$ and $d_n(a) = 10a$ for all $n \in \mathbb{Z}$. Prove that $H_*(U) = 0$ but that that the identity map $\mathrm{id}\colon U_* \to U_*$ is not chain homotopic to zero.

Solution: First, it is clear that

$$B_n(U) = \{10a \mid a \in \mathbb{Z}/100\} = \{0 + 100\mathbb{Z}, 10 + 100\mathbb{Z}, 20 + 100\mathbb{Z}, \ldots, 90 + 100\mathbb{Z}\}.$$

On the other hand, for $a = i + 100\mathbb{Z} \in U_n$ we have $d(a) = 10i + 100\mathbb{Z}$ so $d(a) = 0$ iff $10i = 0 \pmod{100}$ iff $i = 0 \pmod{10}$; this makes it clear that $Z_n(U) = B_n(U)$, and thus that $H_n(U) = 0$.

Now suppose we have maps $s_k\colon U_k \to U_{k+1}$ giving a chain homotopy from id to 0, so $s(d(a)) + d(s(a)) = \mathrm{id}(a) - 0 = a$ for all $a \in U_k$. Put $m_k = s_k(1) \in U_{k+1} = \mathbb{Z}/100$. We must then have $s_k(a) = m_k a$ for all $a \in U_k$. It follows that for $a = 10 \in U_k$ we have

$$10 = d_{k+1}(s_k(10)) + s_{k-1}(d_k(10)) = d_{k+1}(10m_k) + s_{k-1}(100) = 100m_k + 100m_{k-1}.$$

As we are working in $\mathbb{Z}/100$, the right hand side is zero. Thus, the above equation says that $10 = 0$ in $\mathbb{Z}/100$, which is false. This contradiction shows that no such chain homotopy can exist.

Exercise 3. Let U_* be a chain complex in which all the groups U_k are finite-dimensional vector spaces over $\mathbb{Q}$, and all the differentials $d\colon U_k \to U_{k-1}$ are $\mathbb{Q}$-linear.

(a) For each n, choose a basis $b_{n,1}, \ldots, b_{n,p(n)}$ for $B_n(U)$.

(b) Explain why we can choose elements $v_{n+1,1}, \ldots, v_{n+1,p(n)} \in U_{n+1}$ such that $d(v_{n+1,k}) = b_{n,k}$ for all k.

(c) Explain why we can choose additional elements $h_{n,1}, \ldots, h_{n,q(n)} \in Z_n(U)$ such that $b_{n,1}, \ldots, b_{n,p(n)}, h_{n,1}, \ldots, h_{n,q(n)}$ is a basis for $Z_n(U)$. Describe $H_n(U)$ in terms of this basis.

(d) Explain why the list $v_{n,1}, \ldots, v_{n,p(n-1)}, b_{n,1}, \ldots, b_{n,p(n)}, h_{n,1}, \ldots, h_{n,q(n)}$ is a basis for U_n.

(e) Put $V_n = \mathrm{span}(h_{n,1}, \ldots, h_{n,q(n)})$, and consider this as a chain complex with $d = 0$. Construct an injective chain map $i\colon V_* \to U_*$ and a surjective chain map $r\colon U_* \to V_*$.

(f) Define $s\colon U_n \to U_{n+1}$ by $s(b_{n,i}) = v_{n+1,i}$ and $s(v_{n,i}) = 0$ and $s(h_{n,i}) = 0$. Use this to show that U_* is chain homotopy equivalent to V_*.

Solution:

(b) Here we need only note that the elements $b_{n,i}$ lie in $B_n(U)$, which is defined to be the image of the map $d\colon U_{n+1} \to U_n$, so we can choose elements $v_{n+1,i}$ with $d(v_{n+1,i}) = b_{n,i}$.

(c) Here is a standard result from linear algebra:

> Let M be a finite-dimensional vector space, let N be a subspace, and let $n_1, \dots, n_p$ be a basis for N. Then there exist elements $m_1, \dots, m_q \in M$ such that the combined list $n_1, \dots, n_p, m_1, \dots, m_q$ is a basis for M.

We can apply this to the case where $M = Z_n(U)$ and $N = B_n(U)$. This gives a list $h_{n,1}, \dots, h_{n,q(n)} \in Z_n(U)$ such that the combined list of b's and h's is a basis for $Z_n(U)$. This means that the list of h's is a basis for a subspace $V_n \leq Z_n(U)$ with $Z_n(U) = B_n(U) \oplus V_n$. This gives

$$H_n(U) = Z_n(U)/B_n(U) \simeq (B_n(U) \oplus V_n)/B_n(U) \simeq V_n.$$

More explicitly, this means that if we put $c_{ni} = [h_{ni}] = h_{ni} + B_n(U)$ then the list $c_{n,1}, \dots, c_{n,q(n)}$ is a basis for $H_n(U)$.

(d) Consider an element $x \in U_n$. We then have $d(x) \in B_{n-1}(U)$, and the list $b_{n-1,1}, \dots, b_{n-1,p(n-1)}$ is a basis for B_{n-1}, so there is a unique family of coefficients λ_i with $d(x) = \sum_i \lambda_i b_{n-1,i}$. As $d(v_{n,i}) = b_{n-1,i}$, we see that the element $x' = x - \sum_i \lambda_i v_{n,i}$ satisfies $d(x') = 0$, so $x' \in Z_n(U)$. As the list in (c) is a basis for $Z_n(U)$, we see that there are unique families of coefficients μ_j and ν_k such that $x' = \sum_j \mu_j b_{n,j} + \sum_k \nu_k h_{n,k}$, or equivalently $x = \sum_i \lambda_i v_{n,i} + \sum_j \mu_j b_{n,j} + \sum_k \nu_k h_{n,k}$. As all of these coefficients are unique, we see that the indicated list is indeed a basis for U_n.

(e) We can define $i \colon V_n \to U_n$ to be the inclusion, so $i(h_{n,k}) = h_{n,k}$ for $k = 1, \dots, q(n)$. As $h_{n,k} \in Z_n(U)$ by construction we see that $d(h_{n,k}) = 0$ and so $di = 0 = id$. This means that i is a chain map (and it is clearly injective). Next, because the list in (d) is a basis, we can define $r \colon U_n \to V_n$ by $r(v_{n,i}) = 0$ and $r(b_{n,j}) = 0$ and $r(h_{n,k}) = h_{n,k}$. We find that $r(d(v_{n,i})) = r(b_{n-1,i}) = 0 = d(r(v_{n,i}))$ and $r(d(b_{n,j})) = r(0) = 0 = d(r(b_{n,j}))$ and $r(d(h_{n,k})) = r(0) = 0 = d(r(h_{n,k}))$, so r is a chain map (which is clearly surjective).

(f) It is clear that $r \circ i = \mathrm{id} \colon V_* \to V_*$. Next, we have

$$(ds + sd)(v_{n,i}) = d(0) + s(b_{n-1,i}) = v_{n,i} \qquad (\mathrm{id} - ir)(v_{n,i}) = v_{n,i} - i(0) = v_{n,i}$$
$$(ds + sd)(b_{n,j}) = d(v_{n+1,j}) + s(0) = b_{n,j} \qquad (\mathrm{id} - ir)(b_{n,j}) = b_{n,j} - i(0) = b_{n,j}$$
$$(ds + sd)(h_{n,k}) = d(0) + s(0) = 0 \qquad (\mathrm{id} - ir)(h_{n,k}) = h_{n,k} - i(h_{n,k}) = 0.$$

This shows that $ds + sd = \mathrm{id} - ir$, so s gives a chain homotopy between id and ir. This implies that r is a chain homotopy inverse for i, as required.

Exercise 4. Consider the following matrices:

$$D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplication by D gives a homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^3$, and similarly for the other three matrices.

(a) Show that the sequence

$$A_* = (\mathbb{Z}^3 \xleftarrow{D} \mathbb{Z}^3 \xleftarrow{T} \mathbb{Z}^3 \xleftarrow{D} \mathbb{Z}^3 \xleftarrow{T} \mathbb{Z}^3 \leftarrow \cdots)$$

is a chain complex.

(b) Find $UT + DV$ and $TU + VD$.

(c) Use (b) to construct a chain homotopy between certain maps $A_* \to A_*$.

(d) Use this to calculate $H_* A$.

Solution:

(a) We just need to check that the composites of adjacent differentials are zero, or equivalently that the matrix products DT and TD are zero. This is a straightforward calculation.

(b) It is also straightforward to calculate

$$UT = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad TU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad VD = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad DV = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this we get $UT + DV = TU + VD = I$.

(c) Let s be the following system of maps $s_k \colon A_k \to A_{k+1}$:

$$\mathbb{Z}^3 \xrightarrow{V} \mathbb{Z}^3 \xrightarrow{U} \mathbb{Z}^3 \xrightarrow{V} \mathbb{Z}^3 \xrightarrow{U} \mathbb{Z}^3 \xrightarrow{V} \cdots$$

In strictly positive odd degrees we have $ds + sd = TU + VD = I$. In strictly positive even degrees we have $ds + sd = DV + UT = I$. In degree zero we have $ds + sd = DV = I - UT$. Thus, if we define $f_0 = UT$ and $f_k = 0$ for $k \neq 0$, we find that $ds + sd = 1 - f$, so f is chain homotopic to the identity.

(d) Part (c) tells us that the identity map of $H_k A$ is the same as f_*, and so is zero for $k > 0$, so $H_k A = 0$ for $k > 0$. It is also clear that $H_k A = 0$ for $k < 0$. From the definitions we have $H_0 A = \mathbb{Z}^3 / \operatorname{img}(D)$. The vectors $u_1 = (1, 0, 0)$ and $u_2 = De_1 = (1, 0, -1)$ and $u_3 = De_2 = (-1, 1, 0)$ are easily seen to give a basis of $\mathbb{Z}^3$. We also have $De_3 = (0, -1, 1) = -u_2 - u_3$, so $\operatorname{img}(D)$ is spanned by u_2 and u_3, so $H_0 A$ is a copy of $\mathbb{Z}$ generated by $[u_1]$.