MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 12 — Solutions

Please hand in Exercises 1 and 2 by the Wednesday lecture of Week 6. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. We define groups U_k, V_k and W_k (for all $k \in \mathbb{Z}$) and maps between them as follows:

- U_k is a copy of $\mathbb{Z}/4$ with generator u_k, and V_k is a copy of $\mathbb{Z}/16$ with generator v_k, and W_k is a copy of $\mathbb{Z}/4$ with generator w_k.
- The maps $d\colon U_k \to U_{k-1}$ and $d\colon V_k \to V_{k-1}$ and $d\colon W_k \to W_{k-1}$ are given by $d(u_k) = 0$ and $d(w_k) = 0$ and $d(v_k) = 8v_{k-1}$.
- The map $i\colon U_k \to V_k$ is given by $i(u_k) = 4v_k$.
- The map $p\colon V_k \to W_k$ is given by $p(v_k) = w_k$.

(a) Prove that this makes U_*, V_* and W_* into chain complexes.
(b) Prove that i and p are chain maps.
(c) Prove that the sequence $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ is short exact.
(d) Find the homology groups of U_*, V_* and W_*.
(e) Describe the action of the maps i_* and p_* on these homology groups.
(f) By finding suitable snakes, describe the connecting map $\delta\colon H_k(W) \to H_{k-1}(U)$. Check that the resulting long sequence of homology groups is exact.

Note: In working through this problem you will need to refer to various homology classes $[z]$. You must remember that this notation is only meaningful when z is a cycle, i.e. it satisfies $d(z) = 0$. It is easy to violate this rule by accident; you should check your work carefully to ensure that you have not done so.

Solution:

(a) We just need to check the condition $d^2 = 0$. For U and W we already have $d = 0$. For v we have $d^2(v_k) = d(8v_{k-1}) = 64v_{k-2}$ but this is zero because v_{k-2} has order 16.
(b) We have $di(u_k) = d(4v_k) = 32v_{k-1}$ which is again zero because v_{k-1} has order 16. On the other hand, we have $id(u_k) = i(0) = 0$, so $di = id$. Next, we have $pd(v_k) = p(8v_{k-1}) = 8w_{k-1}$, which is zero as w_{k-1} has order 4. On the other hand, we have $d = 0$ on W so $dp(v_k) = 0$ as well. This shows that $dp = pd$, so both i and p are chain maps.
(c) We have $i(m\,u_k) = 4m\,v_k$, so $i(m\,u_k) = 0$ iff $4m$ is divisible by 16 iff m is divisible by 4 iff $m\,u_k = 0$. This proves that i is injective. We have $p(m\,v_k) = 0$ iff $m\,w_k = 0$ iff m is divisible by 4, in which case we have $m\,v_k = i(m/4)\,u_k$. Using this we see that $\operatorname{img}(i) = \ker(p)$. Finally, any element of W_k can be written as $m\,w_k$ for some $m \in \mathbb{Z}$, and this is the same as $p(m\,v_k)$, so p is surjective.
(d) As $d = 0$ on U we have $Z_*(U) = U$ and $B_*(U) = 0$ so $H_*(U) = Z_*(U)/B_*(U) \simeq U_*$. We write $a_k = [u_k]$, so $H_k(U)$ is a copy of $\mathbb{Z}/4$ generated by a_k. Similarly, we write $c_k = [w_k]$, so $H_k(W)$ is a copy of $\mathbb{Z}/4$ generated by c_k. Next, it is easy to see that $Z_k(V)$ is generated by $2v_k$ but $B_k(V)$ is generated by $8v_k$. Thus, if we put $b_k = [2v_k]$ we find that $4b_k = [8v_k] = [d(v_{k+1})] = 0$ and in fact $H_k(V)$ is a copy of $\mathbb{Z}/4$ generated by b_k. In summary, for all three of our complexes, every homology group is isomorphic to $\mathbb{Z}/4$.
(e) We have $i_*(a_k) = [i(u_k)] = [4v_k] = 2b_k$ and $p_*(b_k) = [p(2v_k)] = [2w_k] = 2c_k$.
(f) Consider the sequence $(c_k, w_k, v_k, 2u_{k-1}, 2a_{k-1})$. The element w_k is a cycle representing the homology class c_k, and $p(v_k) = w_k$, and $d(v_k) = 8v_{k-1} = d(2u_{k-1})$, and $2u_{k-1}$ is a cycle representing the class $2a_{k-1}$. Thus, the above sequence is a snake, showing that $\delta(c_k) = 2a_{k-1}$. It follows that the long (i_*, p_*, δ)-sequence has the form
$$\dots \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \dots,$$
and this is visibly exact, with $\operatorname{img} = \ker = \{0, 2\} \subset \{0, 1, 2, 3\} = \mathbb{Z}/4$ at every stage.

Here are some pitfalls to look out for:

- It is tempting to write various expressions involving $[v_k]$, but this is not meaningful. We are working in the group $H_k(V) = Z_k(V)/B_k(V)$, but $d(v_k) \neq 0$ so $v_k \notin Z_k(V)$ so $[v_k]$ is undefined.
- In particular, it is not correct to rewrite $[2v_k]$ as $2[v_k]$ or $[4v_k]$ as $4[v_k]$ (but it is valid to note that $[4v_k] = 2[2v_k]$).
- It is tempting to say that $[4v_k]$ is divisible by 4 and so counts as zero in the group $H_1(V) \simeq \mathbb{Z}/4$. But again, this relies on writing $[4v_k]$ as $4[v_k]$, which is invalid as we have explained. In fact, to say that the coset $[4v_k] = 4v_k + B_k(V)$ is zero would mean that $4v_k$ lies in the group $B_k(V) = \operatorname{img}(d\colon V_{k+1} \to V_k)$, but it is easy to see that $B_k(V) = \{0, 8v_k\}$ so this is false.

Exercise 2. Let U_* and W_* be chain complexes, and suppose we have maps $f_n \colon W_n \to U_{n-1}$ that satisfy $df = -fd \colon W_n \to U_{n-2}$. Put $V_n = U_n \oplus W_n$ and define $d \colon V_n \to V_{n-1}$ by

$$d(u, w) = (d(u) + f(w),\ d(w)).$$

Define maps $U_n \xrightarrow{i} V_n \xrightarrow{p} W_n$ by $i(u) = (u, 0)$ and $p(u, w) = w$.

(a) Prove that V_* is a chain complex.

(b) Prove that i and p are chain maps and that the sequence $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ is short exact.

(c) Prove that the resulting map $\delta \colon H_n(W) \to H_{n-1}(U)$ satisfies $\delta([w]) = [f(w)]$.

Solution:

(a) For $(u, w) \in V_n$ we have $d(u, w) = (d(u) + f(w), d(w))$, and for $(u', w') \in V_{n-1}$ we have $d(u', w') = (d(u') + f(w'), d(w'))$. By taking $u' = d(u) + f(w)$ and $w' = d(w)$, and using $df = -fd$, we see that

$$\begin{aligned} d^2(u, w) &= d(d(u) + f(w), d(w)) \\ &= (d(d(u) + f(w)) + f(d(w)), d(d(w))) = (0 + d(f(w)) + f(d(w)), 0) = (0, 0). \end{aligned}$$

This proves that $d^2 = 0$ on V_*, so V_* is a chain complex.

(b) We now note that

$$\begin{aligned} i(d(u)) &= (d(u), 0) = (d(u) + f(0), d(0)) = d(u, 0) = d(i(u)) \\ p(d(u, w)) &= p(d(u) + f(w), d(w)) = d(w) = d(p(u, w)), \end{aligned}$$

so i and p are chain maps. It is clear that $\mathrm{img}(i) = U_* \oplus 0 = \ker(p)$, so the (i, p) sequence is short exact.

(c) Suppose we have a homology class $\overline{w} = [w] \in H_n(W)$, so $d(w) = 0$. Put $v = (0, w) \in V_n$ and $u = f(w) \in U_{n-1}$. As $df = -fd$ we see that $d(u) = -f(d(w)) = -f(0) = 0$, so we have a well-defined element $\overline{u} = [u] \in H_{n-1}(U)$. Now $p(v) = w$ and $d(v) = (d(0) + f(w), d(w)) = (f(w), 0) = i(u)$. This proves that the list $(\overline{w}, w, v, u, \overline{u})$ is a snake, so $\delta(\overline{w}) = \overline{u}$. By unwinding the notation, we can rewrite this as $\delta([w]) = [f(w)]$, as claimed.

Exercise 3. Let $U_* \xrightarrow{i} V_* \xrightarrow{j} W_*$ be a short exact sequence of chain complexes and chain maps. Suppose that the groups $H_n(U)$ and $H_n(W)$ are finite for all n, and are zero when n is odd. Prove that $H_n(V)$ is finite for all n, with $|H_n(V)| = |H_n(U)||H_n(W)|$.

Solution: When n is odd we have an exact sequence

$$0 = H_n(U) \xrightarrow{i_*} H_n(V) \xrightarrow{p_*} H_n(W) = 0.$$

As $\mathrm{img}(i_*) = 0$ and $\ker(p_*) = H_n(V)$ we see that $H_n(V) = 0$, so $|H_n(U)| = |H_n(V)| = |H_n(W)| = 1$ and the relation $|H_n(V)| = |H_n(U)||H_n(W)|$ is trivially satisfied.

Suppose instead that n is even, so $n - 1$ and $n + 1$ are odd. We then have an exact sequence

$$0 = H_{n+1}(W) \xrightarrow{\delta} H_n(U) \xrightarrow{i_*} H_n(V) \xrightarrow{p_*} H_n(W) \xrightarrow{\delta} H_{n-1}(U) = 0,$$

or in other words a short exact sequence $H_n(U) \to H_n(V) \to H_n(W)$. It follows by Lemma 12.20 that $H_n(V)$ is finite with $|H_n(V)| = |H_n(U)||H_n(W)|$.

Exercise 4. Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain maps between chain complexes. Suppose that for every $w \in W_k$ with $dw = 0$ there exists $v \in V_k$ with $dv = 0$ and $pv = w$. Prove that the sequence $H_*(U) \xrightarrow{i_*} H_*(V) \xrightarrow{p_*} H_*(W)$ is short exact.

Solution: Consider an element $c \in H_k(W)$. Choose a representing cycle $w \in Z_k(W)$. By the assumption in the question, we can choose $v \in V_k$ with $pv = w$ and $dv = 0$. In other words, the element $0 \in U_{k-1}$ satisfies $i(0) = d(v)$. It follows that the sequence $(c, w, v, 0, 0)$ is a snake, so $\delta(c) = 0$. As c was arbitrary, the homomorphism $\delta \colon H_k(W) \to H_{k-1}(U)$ is zero for all k. We know already that the sequence

$$H_{k+1}(W) \xrightarrow{\delta} H_k(U) \xrightarrow{i_*} H_k(V) \xrightarrow{p_*} H_k(W) \xrightarrow{\delta} H_{k-1}(U)$$

is exact. As $\delta = 0$, it follows that i_* is injective and p_* is surjective. This means that the sequence $H_k(U) \xrightarrow{i_*} H_k(V) \xrightarrow{p_*} H_k(W)$ is short exact, as claimed.