MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 14 — Solutions

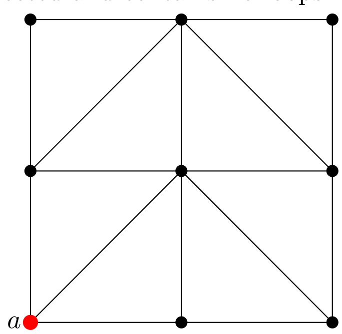
Please hand in exercise 1 by the end of Week 8.

Exercise 1. Let X be a graph, consisting of some points in $\mathbb{R}^2$ (called vertices) and straight edges between them. We assume that no two edges intersect except at the endpoints. In this exercise we will work through the standard calculation of $H_*(X)$.

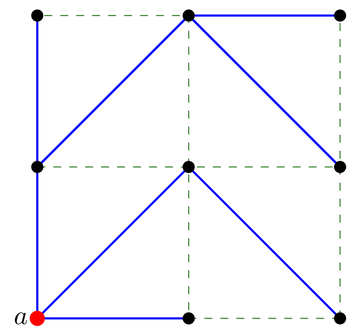
Part (a) below shows an example. However, you should give answers that work for any X, except in cases where the question specifically tells you to use the example in (a).

By a *combinatorial path* in X we mean a sequence of vertices $u_0, \dots, u_r$ such that each pair (u_i, u_{i+1}) is an edge of X. The *combinatorial distance* between vertices a and b is the minimum possible length of a combinatorial path between them.

(a) A *spanning tree* is a subgraph $T \subseteq X$ that contains all of the vertices and some of the edges, with the property that it is connected and contains no loops.



X



T

Show that there always exists a spanning tree. (Just choose a connected loop-free subgraph containing a with as many edges as possible, and prove that it must be a spanning tree.) For the rest of this exercise, we choose a spanning tree T and a vertex $a \in T$.

(b) Show that if x is a vertex of X, then there is a unique combinatorial path u_x that goes from x to a without visiting any vertex twice. Draw some examples of paths u_x in the complex illustrated above.

(c) We also write u_x for the sum of the edges in u_x, considered as an element of $C_1(X)$. What is $\partial(u_x)$?

(d) Define $r(x)$ to be the first vertex on u_x after x (to be interpreted as $r(a) = a$ in the exceptional case where $x = a$). In other words, $r(x)$ is the vertex that we reach after taking one step towards a from x. Annotate the above diagram to show the effect of the map r.

(e) Let e be an edge of T. Show that there is a vertex x such that the endpoints of e are x and $r(x)$.

(f) Part (d) defined r as a map $\text{vert}(T) \to \text{vert}(T)$, where $\text{vert}(T)$ is the set of vertices of T. Explain how to extend this to give a map $r\colon T \to T$. Show that r is homotopic to the identity (but not by a linear homotopy). Deduce that T is contractible.

(g) Now let the edges not in T be $e_1, \dots, e_m$, where $e_q = (x_q, y_q)$. Define $a_q, b_q, c_q \in e_q$ by

$$a_q = \tfrac{3}{4}x_q + \tfrac{1}{4}y_q \qquad b_q = \tfrac{1}{2}x_q + \tfrac{1}{2}y_q \qquad c_q = \tfrac{1}{4}x_q + \tfrac{3}{4}y_q$$

Put $z_q = \langle x_q, y_q \rangle - u_{x_q} + u_{y_q} \in C_1(X)$. Prove that $\partial(z_q) = 0$ (so we have a corresponding element $h_q = [z_q] \in H_1(X)$).

(h) Put $U = X \setminus T$, so U consists of the edges e_q with the endpoints removed. Put $V = X \setminus \{b_1, \dots, b_m\}$. Describe the homology of U, V and $U \cap V$ in terms of the points a_q, b_q, c_q and a.

(i) Use the Mayer-Vietoris sequence to show that $H_1(X) \simeq \mathbb{Z}^m$.

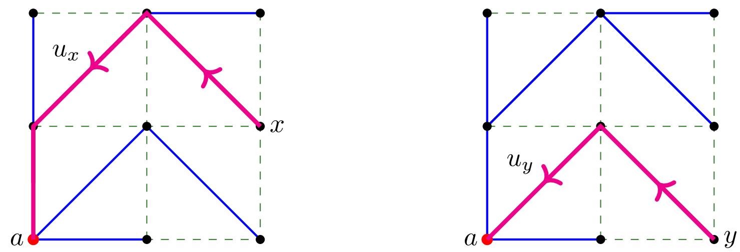
(j) In the construction of the Mayer-Vietoris sequence we use the subcomplex $C_*(U, V) = C_*(U) + C_*(V) \leq C_*(X)$. Show that $z_q \notin C_*(U, V)$. Find elements $z'_q \in C_1(U)$ and $z''_q \in C_1(V)$ such that $\text{sd}^2(z_q) = z'_q + z''_q$, proving that $\text{sd}^2(z_q) \in C_1(U, V)$. (For this you will need to think about $\text{sd}^2(u_x)$ and $\text{sd}^2(\langle x_q, y_q \rangle)$. You can just leave $\text{sd}^2(u_x)$ as $\text{sd}^2(u_x)$ but you will need to analyse $\text{sd}^2(\langle x_q, y_q \rangle)$ in more detail.)

(k) Use z'_q and z''_q to find a snake involving $\text{sd}^2(z_q)$ and thus compute $\delta(h_q)$ in the Mayer-Vietoris sequence. Conclude that the elements $h_1, \dots, h_m$ give a basis for $H_1(X)$.

Solution:

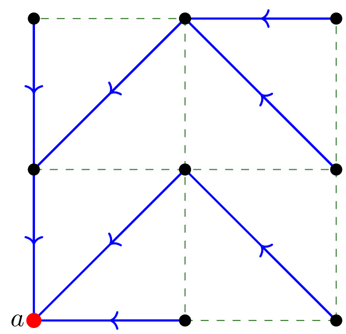
(a) There is at least one connected loop-free subcomplex containing a, namely $\{a\}$. We can list all possible subcomplexes with these properties, and choose one that has as many edges as possible, say T. We claim that T includes every vertex x. Indeed, as X is connected, we can choose a combinatorial path $x = u_0, u_1, \dots, u_r = a$ from x to a. As $u_r = a$ we have $u_r \in T$. Let i be the first index such that $u_i \in T$. If $i = 0$ then $x \in T$ as required. Otherwise we have an edge $e = (u_{i-1}, u_i)$ with one end in T and the other end not in T, so we can add e to T without creating any loops or making it disconnected. This contradicts the assumed maximality of T, so we must have $x \in T$ after all. This proves that T is a spanning tree.

(b) As T is connected and contains all vertices, we can certainly choose a combinatorial path from x to a in T. If we choose such a path of minimum possible length, then it cannot visit any vertex twice (because then we could remove the segment between two visits to get a shorter path). Now suppose that v and w are two different non-repeating paths from x to a in T. They must eventually meet at a; let y be the first place where they meet. We can go along v from x to y, then backwards along w from y to x. This gives a loop in T, contrary to assumption. Thus, the non-repeating path from x to a in T is unique, and can be denoted by u_x.



(c) If the vertices in u_x are $x = v_0, v_1, \dots, v_r = a$ then the corresponding chain is $u_x = \sum_{i=1}^{r} \langle v_{i-1}, v_i \rangle$ giving $\partial(u_x) = \sum_{i=1}^{r} (v_i - v_{i-1}) = v_r - v_0 = a - x$.

(d) Each arrow runs from a vertex x to $r(x)$.



(e) Let the endpoints of e be x and y. After exchanging x and y if necessary, we can assume that the length of u_y is less than or equal to the length of u_x. Let v be the path from x to a consisting of e followed by u_y. If x occurred in u_y, then the section of u_y starting at x would be a combinatorial path from x to a shorter than u_x, which is impossible. Thus, x cannot occur in u_y, so the path v has no repeats, so v must be the same as u_x. However, v starts with (x, y) whereas e_x starts with $(x, r(x))$ so we must have $r(x) = y$.

(f) If $u \in T$ is not a vertex then it lies in the interior of an edge, which has endpoints x and $r(x)$ say by part (e). This means that $u = (1-t)x + t\,r(x)$ for some t with $0 < t < 1$. From the definition of r it is clear that either $r(x) = r^2(x) = a$, or there is an edge of T with endpoints $\{r(x), r^2(x)\}$. We can therefore define $r(u) = (1-t)\,r(x) + t\,r^2(x) \in T$. The formula

$$r((1-t)x + t\,r(x)) = (1-t)\,r(x) + t\,r^2(x)$$

is clearly also valid when $t = 0$ or $t = 1$, so the map $r\colon T \to T$ is continuous on each edge of T. It is therefore continuous on all of T by closed patching.

If the edges $(x, r(x))$ and $(r(x), r^2(x))$ do not point in the same direction, then the line segment joining u to $r(u)$ will not lie within T. Thus, we do not have a linear homotopy between r and the identity. Nonetheless, the map r is still homotopic to the identity: this should be visually clear, and the following definition gives a formal proof:

$$h(s, (1-t)x + t\,r(x)) = \begin{cases} (1-t-s)x + (t+s)\,r(x) & \text{if } 0 \le s \le 1-t \\ (2-t-s)r(x) + (t+s-1)\,r^2(x) & \text{if } 1-t \le s \le 1. \end{cases}$$

(Alternatively, we can define $r_k \colon \text{vert}(T) \to \text{vert}(T)$ by

$$r_k(x) = \begin{cases} x & \text{if } \text{len}(u_x) < k \\ r(x) & \text{if } \text{len}(u_x) \geq k. \end{cases}$$

We can then define $r_k((1-t)x + t\,r(x)) = (1-t)r_k(x) + t\,r_k(r(x))$. We then find that $r_0 = \text{id}$ and r_k is linearly homotopic to r_{k+1} and that $r_k = r$ when k is large.) It follows that r^n is also homotopic to the identity for all n. However, when n is sufficiently large, r^n is the constant map with value a. It follows that T is contractible.

(g) We saw in (c) that $\partial(u_x) = x - a$. It follows that

$$\partial(z_q) = \partial(\langle x_q, y_q \rangle) - \partial(u_{x_q}) + \partial(u_{y_q}) = (y_q - x_q) - (a - x_q) + (a - y_q) = 0.$$

(h) Put $U_q = e_q \setminus \{x_q, y_q\}$. Then U is the disjoint union of the spaces U_q, and each U_q is contractible, and U_q is the path component of b_q. It follows that $H_0(U) = \mathbb{Z}\{[b_1], \dots, [b_m]\}$, and all other homology groups are trivual.

Similarly, the space $U \cap V$ is the disjoint union of the sets $U_q \setminus \{b_q\}$. Moreover, $U_q \setminus \{b_q\}$ is the disjoint union of two open intervals, one of which is the component of a_q, and the other is the component of c_q. It follows that

$$H_0(U \cap V) = \mathbb{Z}\{[a_1], \dots, [a_m], [c_1], \dots, [c_m]\},$$

and all other homology groups are trivial. We also know that X is connected so $H_0(X) = \mathbb{Z}[a]$ but we do not yet know about the other homology groups.

Finally, the set V consists of T with some half-open intervals attached by their endpoints. We can shrink these intervals back to their endpoints, so V is homotopy equivalent to T and so is contractible. It follows that $H_0(V) = \mathbb{Z}.[a]$ and all other homology groups are trivial.

(i) We have inclusion maps

$$\begin{array}{ccc} U \cap V & \xrightarrow{i} & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow[l]{} & X, \end{array}$$

and the associated maps on H_0 are given by

$$i_*([a_q]) = i_*([c_q]) = [b_q]$$
$$j_*([a_q]) = j_*([c_q]) = [a]$$
$$k_*([b_q]) = l_*([a]) = [a].$$

The only interesting part of the Mayer-Vietoris sequence is

$$0 = H_1(U) \oplus H_1(V) \to H_1(X) \xrightarrow{\delta} H_1(U \cap V) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_1(U) \oplus H_1(V).$$

The kernel of the map $\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]$ is easily seen to be the free abelian group generated by the elements $c_q - a_q$ (for $1 \leq q \leq m$). Exactness means that δ gives an isomorphism from $H_1(X)$ to this kernel. Thus, there is a unique element $w_q \in H_1(X)$ with $\delta(w_q) = c_q - a_q$, and the elements $w_1, \dots, w_m$ give a basis for $H_1(X)$ over $\mathbb{Z}$. The Mayer-Vietoris sequence also shows easily that $H_j(X) = 0$ for $j > 1$.

(j) The chain z_q involves $\langle x_q, y_q \rangle$. This contains the point x_q which is not in U, and also the point b_q which is not in V, so $\langle x_q, y_q \rangle$ is not in $S_1(U) \cup S_1(V)$, so z_q is not in $C_1(U, V)$. On the other hand, we have

$$\text{sd}(\langle x_q, y_q \rangle) = \langle b_q, y_q \rangle - \langle b_q, x_q \rangle$$
$$\text{sd}^2(\langle x_q, y_q \rangle) = \langle c_q, y_q \rangle - \langle c_q, b_q \rangle - \langle a_q, x_q \rangle + \langle a_q, b_q \rangle.$$

We can thus put

$$z'_q = -\langle c_q, b_q \rangle + \langle a_q, b_q \rangle$$
$$z''_q = \langle c_q, y_q \rangle - \langle a_q, x_q \rangle - \text{sd}^2(u_{x_q}) + \text{sd}^2(u_{y_q}).$$

We find that $z'_q \in C_1(U)$ and $z''_q \in C_1(V)$ and $z'_q + z''_q = \text{sd}^2(z_q)$ as required.

(k) It is easy to see that $\partial(z'_q) = c_q - a_q$. We also have $\partial(\text{sd}^2(z_q)) = \text{sd}^2(\partial(z_q)) = \partial(0) = 0$, so we must have $\partial(z''_q) = a_q - c_q$. Note here that $a_q, c_q \in U \cap V$ so we can regard $c_q - a_q$ as an element of $C_0(U \cap V)$, and it satisfies

$$\left[\begin{smallmatrix} i_\# \\ -j_\# \end{smallmatrix}\right](c_q - a_q) = (c_q - a_q,\ a_q - c_q) = \partial(z'_q,\ z''_q).$$

It follows that the list

$$(h_q,\ \text{sd}^2(z_q),\ (z'_q, z''_q),\ c_q - a_q,\ [c_q] - [a_q])$$

is a snake for the showt exact sequence

$$C_*(U \cap V) \xrightarrow{\left[\begin{smallmatrix} i_\# \\ -j_\# \end{smallmatrix}\right]} C_*(U) \oplus C_*(V) \xrightarrow{[\, k_\# \;\; l_\# \,]} C_*(U, V),$$

proving that $\delta(h_q) = [c_q] - [a_q]$. This means that h_q is the same as the element that we called w_q in part (h), so the list $(h_1, \ldots, h_m)$ is a basis for $H_1(X)$ as claimed.