MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 15 — Solutions

Please hand in exercises 3 and 4 by the Wednesday lecture of Week 10.

Exercise 1. Recall that the Möbius strip can be defined as

$$M = \{(z, w) \in S^1 \times B^2 \mid w^2/z \in [0, 1] \subseteq \mathbb{R} \subseteq \mathbb{C}\}.$$

Show that there is a covering map $p\colon S^1 \times [-1, 1] \to M$.

Solution: We can define $p\colon S^1 \times [-1, 1] \to S^1 \times B^2$ by $p(u, t) = (u^2, ut)$, and it is clear that $p(-u, -t) = p(u, t)$ for all $(u, t) \in S^1 \times [-1, 1]$. If $(z, w) = (u^2, ut)$ then $w^2/z = (ut)^2/u^2 = t^2 \in [0, 1]$, so we see that the image of p is contained in M. For $(z, w) \in M$ we can choose $u \in S^1$ such that $u^2 = z$, and then put $t = w/u$. As $(z, w) \in M$ we have $t^2 = (w/u)^2 = w^2/z \in [0, 1]$ so $t \in [-1, 1]$. From the definition of t we have $ut = w$, and it follows that $p(u, t) = (z, w)$. More generally, for $(v, s) \in S^1 \times [-1, 1]$ we have $p(v, s) = (z, w)$ iff $(v^2, vs) = (u^2, ut)$ and it is easy to see that this holds iff $(v, s) = (-1)^m(u, t)$ for some $m \in \{0, 1\}$. This proves that $|p^{-1}\{(z, w)\}| = 2$ for all $(z, w) \in M$, and so suggests that p is a covering map. To make this more rigorous, put

$$U = \{(a, b) \in M \mid a \neq -z\}.$$

This is clearly an open subset of M containing (z, w). We have $(v, s) \in p^{-1}(U)$ iff $(v/u)^2 \neq -1$. As $v/u \in S^1$, the condition $(v/u)^2 \neq -1$ is equivalent to $\mathrm{Re}(v/u) \neq 0$. We can therefore define $f\colon p^{-1}(U) \to \{0, 1\}$ by

$$f(v, s) = \begin{cases} 0 & \text{if } \mathrm{Re}(v/u) > 0 \\ 1 & \text{if } \mathrm{Re}(v/u) < 0. \end{cases}$$

This is continuous by open patching, and so gives a continuous map $\langle p, f\rangle\colon p^{-1}(U) \to U \times \{0, 1\}$, and one can check that this is in fact a homeomorphism. This proves that we have a covering, as claimed.

Exercise 2. Let $p\colon X \to Y$ be a covering map. Let Y_0 be a subset of Y, and put $X_0 = p^{-1}(Y_0)$, so we have a restricted map $p_0\colon X_0 \to Y_0$. Give X_0 and Y_0 the subspace topologies inherited from X and Y respectively. Prove that p_0 is also a covering.

Solution: Consider a point $y \in Y_0$. As p is a covering, we can choose an open subset $V \subseteq Y$ with $y \in V$, and a continuous map $f\colon p^{-1}(V) \to F$ (for some discrete space F) such that the combined map $\langle p, f\rangle\colon p^{-1}(V) \to V \times F$ is a homeomorphism. Put $V_0 = V \cap Y_0$, so V_0 is open in Y_0 and $y \in V_0$ and $V_0 \subseteq V$. It follows that the set $p_0^{-1}(V_0) = p^{-1}(V_0)$ is a subset of $p^{-1}(V)$, so we can define $f_0\colon p_0^{-1}(V_0) \to F$ to be the restriction of f. This is again continuous (by Proposition 3.28). It follows that the combined map $\langle p_0, f_0\rangle\colon p_0^{-1}(V_0) \to V_0 \times F$ is also continuous. It will be enough to show that this is a homeomorphism.

Now let $m\colon V \times F \to p^{-1}(V)$ be the inverse of $\langle p, f\rangle\colon p^{-1}(V) \to V \times F$, so m is continuous by assumption. For $y \in V_0$ and $s \in F$ we have $\langle p, f\rangle(m(y, s)) = (y, s)$, which means that $p(m(y, s)) = y$ and $f(m(y, s)) = s$. As $p(m(y, s)) = y$ with $y \in V_0$ we see that $m(y, s) \in p^{-1}(V_0) = p_0^{-1}(V_0)$. We can thus define $m_0\colon V_0 \times F \to p_0^{-1}(V_0)$ by $m_0(y, s) = m(y, s)$. This fits in a commutative diagram

$$\begin{array}{ccc} V_0 \times F & \xrightarrow{m_0} & p_0^{-1}(V_0) \\ {\scriptstyle \mathrm{inc}\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{inc}} \\ V \times F & \xrightarrow[m]{} & p^{-1}(V). \end{array}$$

It follows using Proposition 3.28) that m_0 is continuous. From the fact that m is inverse to $\langle p, f\rangle$, it follows easily that m_0 is inverse to $\langle p_0, f_0\rangle$. It follows that $\langle p_0, f_0\rangle$ is a homeomorphism as required.

Exercise 3. Suppose that $p_0\colon X_0 \to Y_0$ and $p_1\colon X_1 \to Y_1$ are covering maps. Define $p = p_0 \times p_1\colon X_0 \times X_1 \to Y_0 \times Y_1$, so $p(x_0, x_1) = (p_0(x_0), p_1(x_1))$. Show that p is a covering map (with respect to the product topologies on $X_0 \times X_1$ and $Y_0 \times Y_1$).

Solution: Consider a point $y = (y_0, y_1) \in Y_0 \times Y_1$. As $p_0\colon X_0 \to Y_0$ is a covering, we can find a discrete space F_0, an open subset $V_0 \subseteq Y_0$ containing Y_0, and a continuous map $f_0\colon p_0^{-1}(V_0) \to F_0$ such that the combined map $\langle p_0, f_0\rangle\colon p_0^{-1}(V_0) \to V_0 \times F_0$ is a homeomorphism. Similarly, as $p_1\colon X_1 \to Y_1$ is a covering, we can find a discrete space F_1, an open subset $V_1 \subseteq Y_1$ containing Y_1, and a continuous map $f_1\colon p_1^{-1}(V_1) \to F_1$ such that the combined map $\langle p_1, f_1\rangle\colon p_1^{-1}(V_1) \to V_1 \times F_1$ is a homeomorphism. It will be convenient to write U_i for $p_i^{-1}(V_i)$ and let $m_i\colon V_i \times F_i \to U_i$ be the inverse of the homeomorphism $\langle p_i, f_i\rangle\colon U_i \to V_i \times F_i$.

Now note that the box $V = V_0 \times V_1$ is an open subset of $Y_0 \times Y_1$ which contains y. Put $U = p^{-1}(V)$, which is an open subset of $X_0 \times X_1$. We claim that $U = U_0 \times U_1$, or equivalently $p^{-1}(V) = p_0^{-1}(V_0) \times p_1^{-1}(V_1)$. Indeed, for a point $x = (x_0, x_1) \in X_0 \times X_1$ we have

$$\begin{aligned} x \in p^{-1}(V) &\Leftrightarrow \text{ the element } p(x) = (p_0(x_0), p_1(x_1)) \text{ lies in the set } V = V_0 \times V_1 \\ &\Leftrightarrow p_0(x_0) \in V_0 \text{ and } p_1(x_1) \in V_1 \\ &\Leftrightarrow x_0 \in p_0^{-1}(V_0) \text{ and } x_1 \in p_1^{-1}(V_1) \\ &\Leftrightarrow x \in p_0^{-1}(V_0) \times p_1^{-1}(V_1), \end{aligned}$$

as required. We can thus define a map

$$f \colon U = U_0 \times U_1 \to F_0 \times F_1$$

by $f(x_0, x_1) = (f_0(x_0), f_1(x_1))$. We can also define a map

$$m \colon V \times F_0 \times F_1 = V_0 \times V_1 \times F_0 \times F_1 \to U_0 \times U_1 = U$$

by $m(y_0, y_1, s_0, s_1) = (m_0(y_0, s_0),\ m_1(y_1, s_1))$. It is easy to see that f and m are continuous and that m is inverse to $\langle p, f\rangle$, so $V_0 \times V_1$ is trivially covered, as required.

Exercise 4. Let $T = S^1 \times S^1$ be the torus, and define $p \colon T \to T$ by $p(u, v) = (u^2, v^2)$. Prove that p is a covering. Put $Y = \{(u, v) \in T \mid u = 1 \text{ or } v = 1\}$ and $X = p^{-1}(Y)$. Draw a picture of X, as a finite collection of points and arcs joining them. Draw a similar picture of Y, and annotate your pictures to illustrate the effect of the covering map $p \colon X \to Y$.

Solution: Consider a point $(x, y) \in T$. Choose $u_1, v_1 \in S^1$ with $u_1^2 = x$ and $v_1^2 = y$. Put

$$V = \{(s, t) \in T \mid s \neq -x \text{ and } t \neq -y\}.$$

For $(u, v) \in T$ we have

$$\begin{aligned} (u, v) \in p^{-1}(V) &\Leftrightarrow p(u, v) \in V \\ &\Leftrightarrow u^2 \neq -x \text{ and } v^2 \neq -y \\ &\Leftrightarrow (u/u_1)^2 \neq -1 \text{ and } (v/v_1)^2 \neq -1 \\ &\Leftrightarrow (u/u_1) \neq \pm i \text{ and } (v/v_1) \neq \pm i \\ &\Leftrightarrow \mathrm{Re}(u/u_1) \neq 0 \text{ and } \mathrm{Re}(v/v_1) \neq 0. \end{aligned}$$

Now define a continuous map $\sigma \colon \mathbb{R} \setminus \{0\} \to \{1, -1\}$ by

$$\sigma(t) = t/|t| = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0. \end{cases}$$

Using this, we can define $f \colon p^{-1}(V) \to \{1, 1\}^2$ by

$$f(u, v) = (\sigma(\mathrm{Re}(u/u_1)),\ \sigma(\mathrm{Re}(u/u_1)).$$

It is then easy to see that the combined map $\langle p, f\rangle \colon p^{-1}(V) \to V \times \{1, -1\}^2$ is a homeomorphism. This proves that p is a covering.

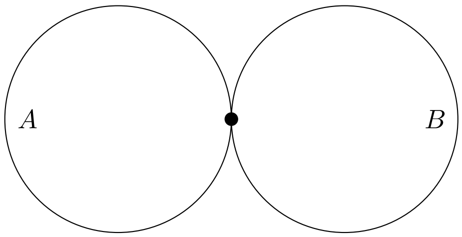
Now put $Y = \{(s, t) \in T \mid s = 1 \text{ or } t = 1\}$ and

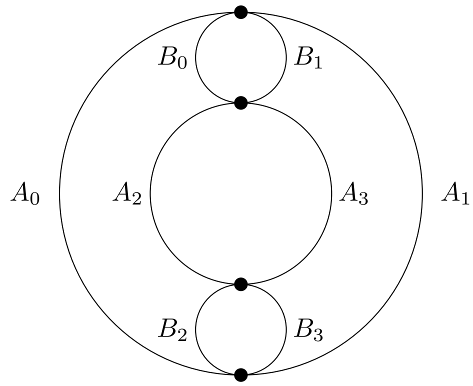
$$\begin{aligned} X = p^{-1}(Y) &= \{(u, v) \in T \mid u^2 = 1 \text{ or } v^2 = 1\} \\ &= \{(u, v) \in T \mid u = 1 \text{ or } u = -1 \text{ or } v = 1 \text{ or } v = -1\}. \end{aligned}$$

Put $A = S^1 \times \{1\}$ and $B = \{1\} \times S^1$, so $Y = A \cup B$. Put

$$\begin{aligned} A_0 &= \{(u, 1) \mid u \in S^1,\ \mathrm{Im}(u) \geq 0\} & A_1 &= \{(u, 1) \mid u \in S^1,\ \mathrm{Im}(u) \leq 0\} \\ A_2 &= \{(u, -1) \mid u \in S^1,\ \mathrm{Im}(u) \geq 0\} & A_3 &= \{(u, -1) \mid u \in S^1,\ \mathrm{Im}(u) \leq 0\} \\ B_0 &= \{(1, v) \mid v \in S^1,\ \mathrm{Im}(v) \geq 0\} & B_1 &= \{(1, v) \mid v \in S^1,\ \mathrm{Im}(v) \leq 0\} \\ B_2 &= \{(-1, v) \mid v \in S^1,\ \mathrm{Im}(v) \geq 0\} & B_3 &= \{(-1, v) \mid v \in S^1,\ \mathrm{Im}(v) \leq 0\}. \end{aligned}$$

Then X is the union of these 8 sets, and p maps each set A_i to A, and each set B_i to B. Everything can be illustrated as follows:





Exercise 5. Let $p\colon X \to Y$ be a 1-sheeted covering. Prove that p is a homeomorphism.

Solution: Because p is a 1-sheeted covering, for each $y \in Y$ we have $|p^{-1}\{y\}| = 1$, so $p^{-1}\{y\} = \{s(y)\}$ for some element $s(y) \in X$. As $s(y) \in p^{-1}\{y\}$ we have $p(s(y)) \in \{y\}$ or equivalently $p(s(y)) = y$. This proves that $p \circ s = \mathrm{id}\colon Y \to Y$. Now suppose we start with $x \in X$ and consider the point $x' = s(p(y))$ As $p \circ s = \mathrm{id}$ we have $p(x') = p(s(p(y))) = p(y)$. This means that $x', y \in p^{-1}\{p(y)\}$, but $|p^{-1}\{p(y)\} = 1$, so we must have $x' = y$. Here x' was defined to be $s(p(y))$, so we conclude that $s(p(y)) = y$, or in other words $s \circ p = \mathrm{id}\colon X \to X$. Form this we see that p is a bijection with inverse s. All that is left is to check that s is continuous.

For this we make the following preliminary claim: if $V \subseteq Y$ is a trivially covered open set, then the map $p\colon p^{-1}(V) \to V$ is a homeomorphism. This is clear if $V = \emptyset$, so suppose instead that $V \neq \emptyset$, and choose a point $a \in V$. As V is trivially covered, we can choose a discrete space F and a continuous map $f\colon p^{-1}(V) \to F$ such that the combined map $\langle p, f\rangle\colon p^{-1}(V) \to V \times F$ is a homeomorphism. It follows that f gives a bijection $p^{-1}\{a\} \to F$, but $|p^{-1}\{a\}| = 1$, so $|F| = 1$. As $|F| = 1$ we see that the projection $V \times F \to V$ is a homeomorphism, and we can compose this with the map $\langle p, f\rangle$ to see that the map $p\colon p^{-1}(V) \to V$ is a homeomorphism as claimed.

Now consider an open set $U \subseteq X$; we need to check that the set $W = s^{-1}(U) = p(U) \subseteq Y$ is open in Y. It will be enough to check that for each $y \in W$ there is an open set V with $y \in V \subseteq W$. As p is a covering we can choose a trivially covered open set V_0 containing y. The set $U_0 = U \cap p^{-1}(V_0)$ is open in $p^{-1}(V_0)$, and the map $p\colon p^{-1}(V_0) \to V_0$ is a homeomorphism, so the set $p(U_0) = p(U) \cap V_0$ is open in V_0. As V_0 is open in Y it follows that $p(U_0)$ is also open in Y (by Lemma 3.30). It is also clear that $y \in p(U_0)$ and $p(U_0) \subseteq p(U) = W$, so we can take $V = p(U_0)$ and this has the required properties.